Chapter 3:

DYNAMICS OF PERFECT INCOMPRESSIBLE FLUIDS

1/ INTRODUCTION

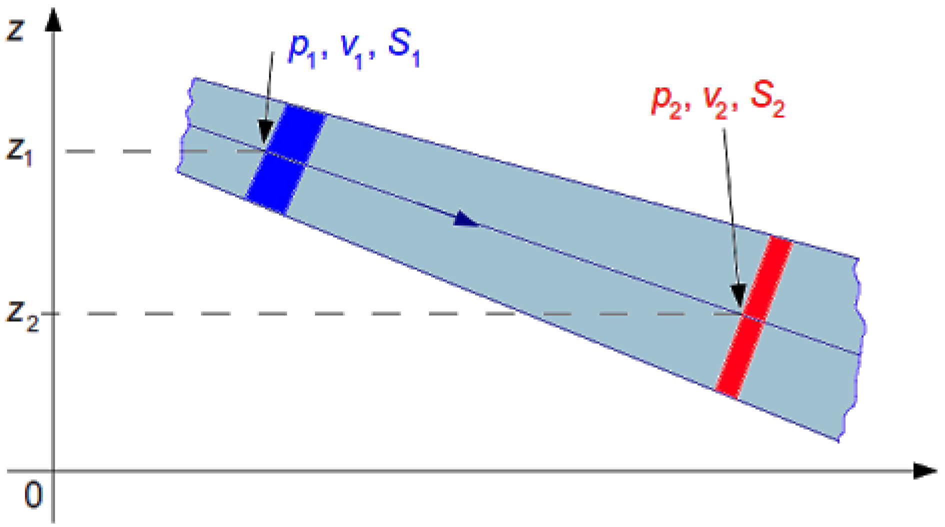
- Dynamics means fluids **in motion** . Unlike solids, moving parts of a fluid can move at different speeds.

- We use the equations that govern the dynamics of perfect incompressible fluids:

- The continuity equation (conservation of mass),
- Bernoulli's theorem (conservation of energy),
- Euler's theorem (conservation of momentum).

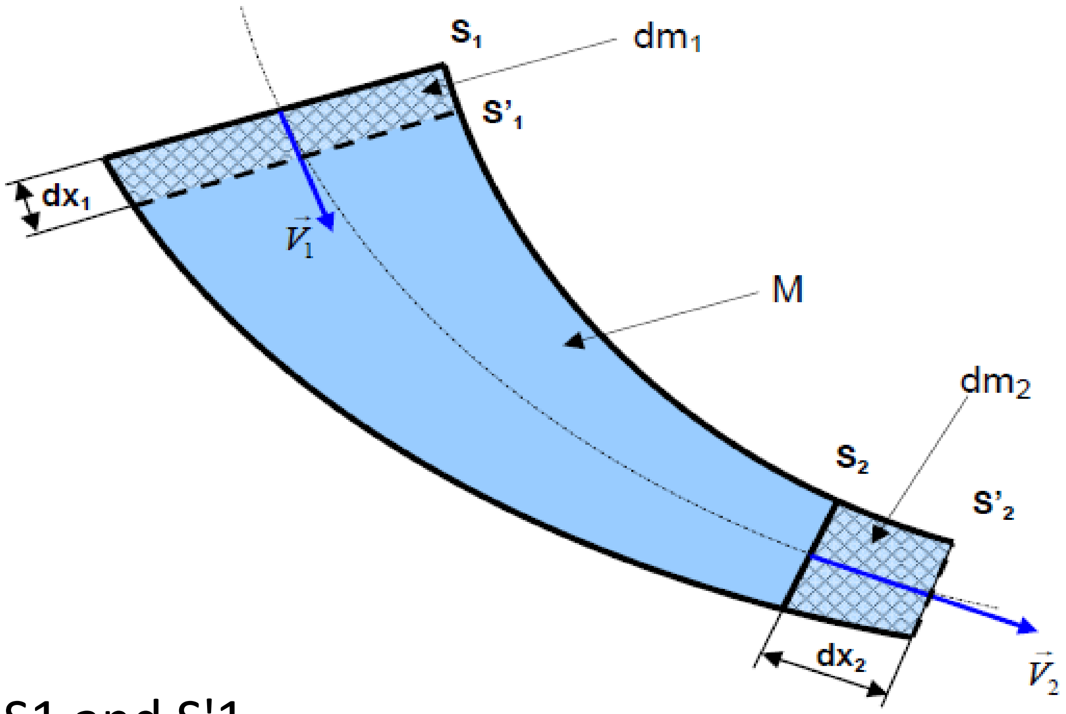
2/ PERMANENT FLOW

The flow of a fluid is said to be permanent if the field of velocity vectors of the fluid particles is constant at time t.



3/ CONTINUITY EQUATION

- Consider a vein of an incompressible fluid
of density ρ animated by a permanent flow.
- **S1 and S2** respectively the input section and
the fluid outlet section at time t,
- S'1 and S'2 respectively the sections
entry and exit of the fluid at time t'=(t+dt),
- ***V1 and V2*** flow velocity vectors.
- **dx1 and dx2 :** respectively the displacements
sections S1 and S2 during the time interval dt ,
- **dm1:** incoming elementary mass between sections S1 and S'1,
- **dm2:** outgoing elementary mass between sections S2 and S'2,
- **dV1 :** incoming elementary volume between sections S1 and S'1,
- **dV2:** outgoing elementary volume between sections S2 and S'2,



We have: $dV_1 = dV_2$ ……………. (1)

Since we have: $\rho = \frac{m}{V}$

$\frac{dm_1}{\rho} = \frac{dm_2}{\rho}$ (because ρ : $constante$)

So: $dm_1 = dm_2$ (Conservation of mass)

According to relation (1):

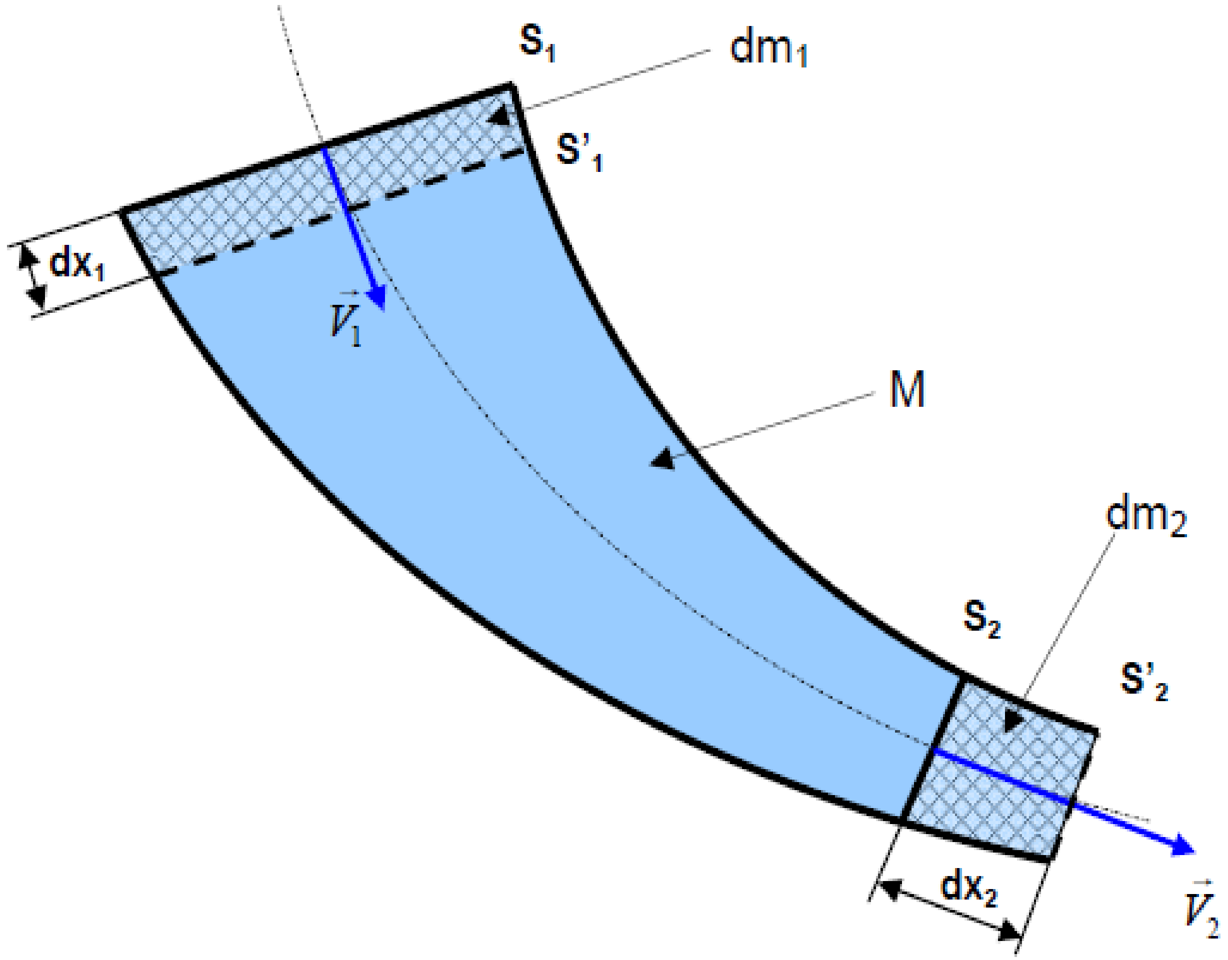
dV1 = dV2

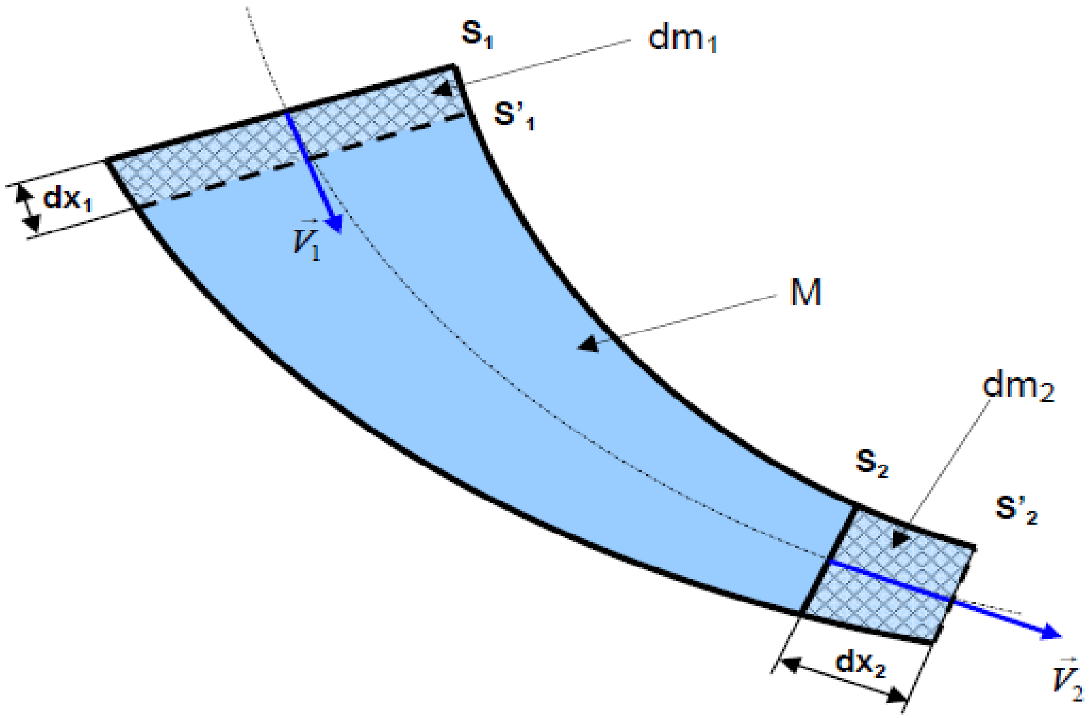
$S1 . dx_1 = S_2 . dx_2$

We divide the equation by dt : $S_1 \frac{dx_1}{dt} = S_2 \frac{dx_2}{dt}$

So, we get:

$S1 . V_1 = S_2 . V_2$ = Constant





$S_1 v_1 = S_2 v_2 = C^{ste}$ is the continuity equation

4/ CONCEPT OF THROUGHPUT

4.1/ Volume flow

Volume flow is the ratio of volume flow per unit time.

$$Q_V = \frac{dV}{dt}$$

noting that: $dV = S.\ dx$ we can also write that:

$$Q_V = \frac{dV}{dT} = \frac{Sdl}{dt} = Sv_{moy}, \quad [m^3/s]$$

- **4.2/ Mass flow**

It is the mass of fluid per unit time:

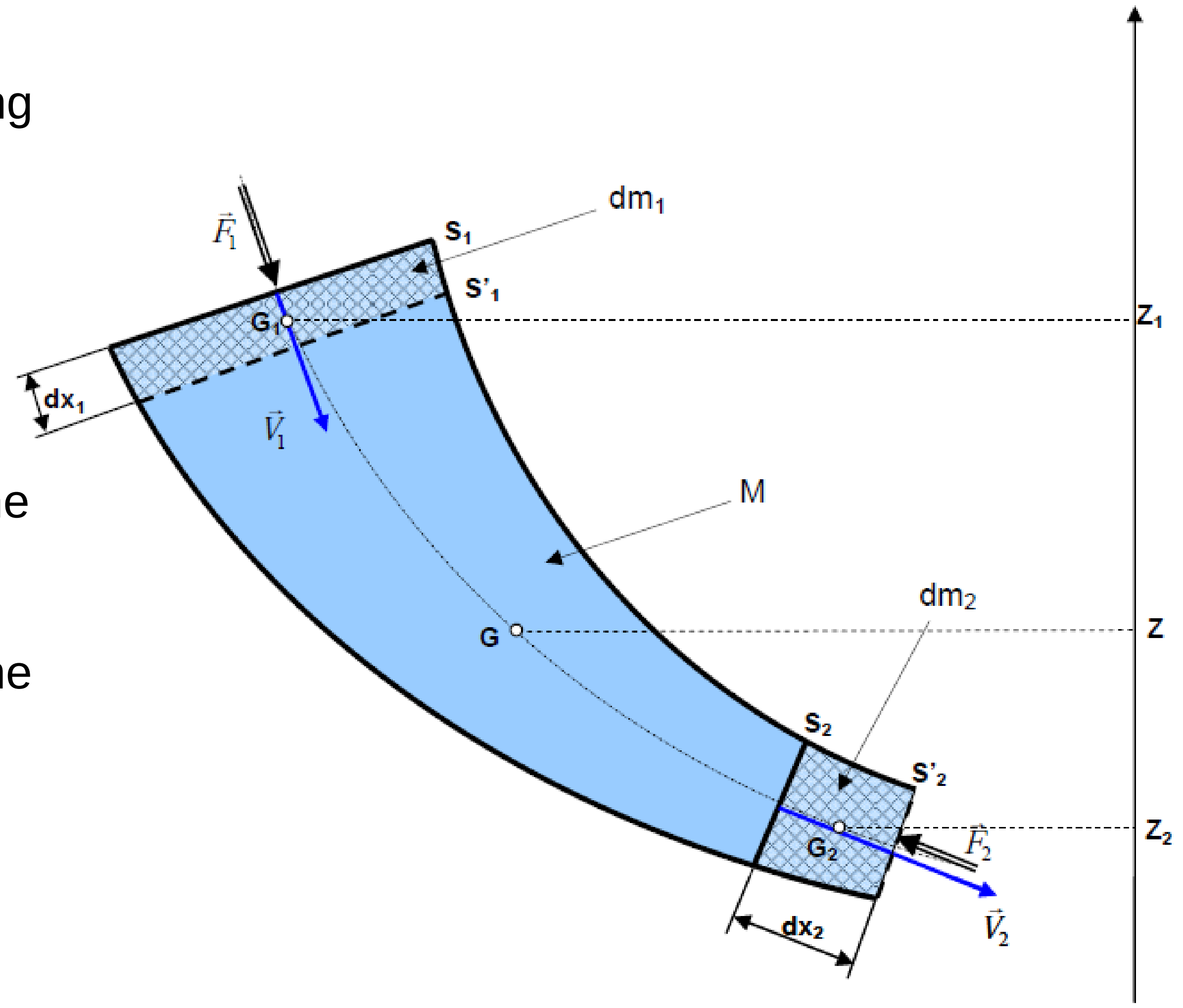
$$Q_m = \frac{dm}{dT} = \frac{\rho dV}{dt} = \rho Q_V \,, \quad [\text{kg/s}]$$ (Relation between Qm and Qv)

5/ BERNOULLI'S THEOREM – CASE OF A FLOW WITHOUT EXCHANGE OF WORK

Let us take again the diagram of the fluid vein of paragraph 3 with the same notations and the following assumptions:

- The fluid is perfect and incompressible.
- The flow is permanent.
- The flow is in a perfectly smooth pipe .

We consider a vertical *Z axis* directed upwards.
We note Z_1, Z_2 and Z respectively the altitudes of the centers of gravity of the masses
dm_1, dm_2 and M.
We denote by F_1 and F_2 respectively the norms of the pressure forces of the
fluid acting at the level of the sections S_1 and S_2.



At time t the mass fluid (dm1 + M) is between S1 and S2. Its mechanical energy is:

$$E_{mec} = E_{pot} + E_{cin} = (dm_1.g.Z_1 + MgZ) + \frac{1}{2}dm_1.V_1^2 + \int_{S'_1}^{S_2}\frac{dm.V^2}{2}$$

At time t'=(t+dt) the fluid of mass (M+dm 2) is between S' 1 and S' 2 . Its mechanical energy is:

$$E'_{mec} = E'_{pot} + E'_{cin} = (MgZ + dm_2.g.Z_2) + \int_{S'_1}^{S_2}\frac{dm.V^2}{2} + \frac{1}{2}dm_2.V_2^2$$

We apply the mechanical energy theorem to the fluid between t and t': “ The variation of the mechanical energy is equal to the sum of the works of the external forces . »

$$E'_{mec} - E_{mec} = W_{\text{Pressure forces}} = F_1.dx_1 - F_2.dx_2 \Leftrightarrow E'_{mec} - E_{mec} = P_1.S_1.dx_1 - P_2.S_2.dx_2 = P_1.dV_1 - P_2.dV_2$$

by simplifying we obtain: $dm_2.g.Z_2 + \frac{1}{2}dm_2.V_2^2 - dm_1.g.Z_1 - \frac{1}{2}.dm_1.V_1^2 = \frac{P_1}{\rho_1}.dm_1 - \frac{P_2}{\rho_2}.dm_2$

By conservation of mass: $dm_1 = dm_2 = dm$ and since the fluid is incompressible: $\rho_1 = \rho_2 = \rho$, We arrive at Bernoulli's equation:

$$\boxed{\frac{V_2^2 - V_1^2}{2} + \frac{P_2 - P_1}{\rho} + g(Z_2 - Z_1) = 0}$$

The unit of each term of the relation (4) is the joule per kilogram (J/kg)

According to relation (4), we can then write:

$$\boxed{\frac{V_2^2}{2}+\frac{P_2}{\rho}+g.z_2=\frac{V_1^2}{2}+\frac{P_1}{\rho}+g.z_1}$$

6/ BERNOULLI'S THEOREM – CASE OF A FLOW WITH EXCHANGE OF WORK

It is further assumed that a hydraulic machine is placed between the sections S_1 and S_2. This machine is characterized by a net power P_{net} exchanged with the fluid, a power on the shaft P_a and a certain efficiency η . This machine can be either a turbine or a pump .

- In the case of a pump: the efficiency is given by the following expression:

$$\eta = \frac{P_{net}}{P_a}$$

- In the case of a turbine: the efficiency is given by the following expression:

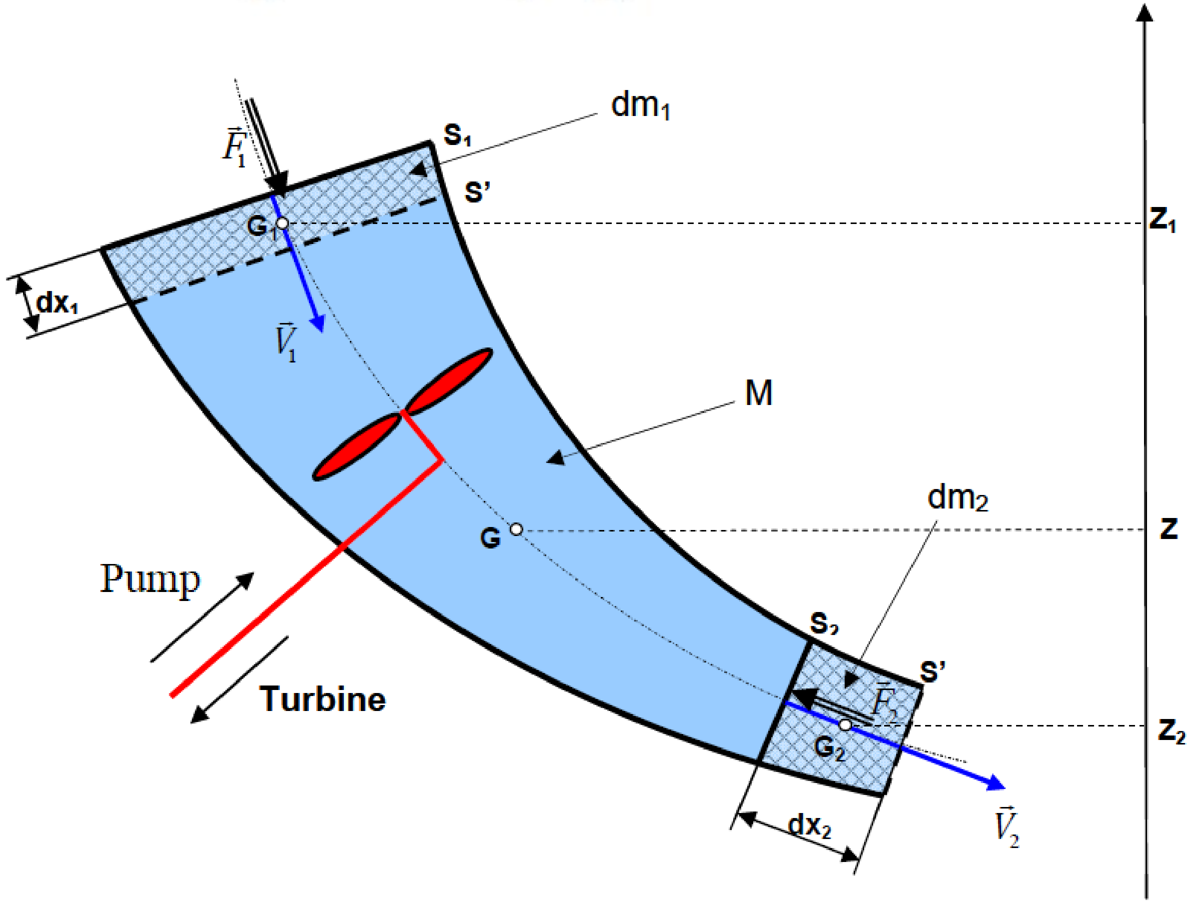
$$\eta = \frac{P_a}{P_{net}}$$

Between times t and t'=(t+dt), the fluid exchanged net work $W_{net} = P_{net}.dt$ with the hydraulic machine. W_{net} is assumed to be positive if it is a pump and negative if it is a turbine.

We denote by F_1 and F2 respectively the standards of the pressure forces of the fluid acting at sections S_1 and S2.

At time t the mass fluid (dm_1 + M) is between S_1 and S2. Its mechanical energy :

$$E_{mec} = E_{pot} + E_{cin} = (dm_1.g.Z_1 + MgZ) + \frac{1}{2}dm_1.V_1^2 + \int_{S'_1}^{S_2}\frac{dm.V^2}{2}$$



At time t'=(t+dt) the mass fluid ($M+dm_2$) is between S'_1 and S'2. Her

mechanical energy is: $E'_{mec} = E'_{pot} + E'_{cin} = (MgZ + dm_2.g.Z_2) + \int_{S'_1}^{S_2} \frac{dm.V^2}{2} + \frac{1}{2} dm_2.V_2^2$

We apply the mechanical energy theorem to the fluid between t and t' :«

The variation of the mechanical energy is equal to the sum of the works of

the external forces. »>, considering this time the work of the hydraulic machine

$$E'_{mec} - E_{mec} = F_1.dx_1 - F_2.dx_2 + P_{net}.dt$$

$E'_{mec} - E_{mec} = P_1.S_1.dx_1 - P_2.S_2.dx_2 + P_{net.}.dt = P_1.dV_1 - P_2.dV_2 + P_{net}.dt$ by simplifying we will have:

$dm_2.g.Z_2 + \frac{1}{2} dm_2.V_2^2 - dm_1.g.Z_1 - \frac{1}{2}.dm_1.V_1^2 = \frac{P_1}{\rho_1}.dm_1 - \frac{P_2}{\rho_2}.dm_2 + P_{net}.dt$ Per conservation

mass: dm_1 = dm, = dm and since the fluid is incompressible: $\rho_1 = \rho_2 = \rho$,

we arrive at Bernoulli's equation : $\boxed{\frac{V_2^2 - V_1^2}{2} + \frac{P_2 - P_1}{\rho} + g(Z_2 - Z_1) = \frac{P_{net}}{q_m}}$ (5)

- <u>Momentum (*EULER'S THEOREM*)</u>

This theorem makes it possible to determine the forces exerted by the moving fluid on the objects which surround them .

- Euler's theorem results from the application of the momentum theorem to the flow of a fluid:

$$\sum \overrightarrow{F}ext = \frac{d\overrightarrow{P}}{dt}$$

with $\overrightarrow{P} = m\overrightarrow{V}_G$: momentum.

$$\boxed{\sum \overline{F}ext = q_m(\overrightarrow{V_2} - \overrightarrow{V_1})}$$