Chapter 2

Sets and applications

This chapter introduces fundamental concepts of sets, elements, subsets, and set operations (union, intersection, complement) that are frequently used in linear algebra, it focuses also on functions and mappings, including injective, surjective, and bijective functions. It explores the composition of functions and the concept of reciprocal functions, which are crucial for understanding transformations and operations. Additionally, it discusses the direct and inverse images of sets under a function, providing students with the essential tools to represent and analyze mathematical relationships and functions. These concepts are fundamental for a wide range of applications in linear algebra, including linear transformations, matrix operations, and system-solving.

2.1 Definitions

Definition 2.1.1. *(Sets in Extension and Comprehension) A set* $E = \{a, b, ...\}$ *defined by listing its elements* $a, b, ...$ *is called a set defined in extension. If* $E = \{x, P(x)\}$ *is the set of elements* x *that satisfy the proposition* P*, then* E *is called a set defined in comprehension.*

Example 2.1.2.

- $\{1, 2\}$ *is a set defined in extension.*
- $\{x \in \mathbb{R},\ x^2 - 2 = 0\}$ *is a set defined in comprehension.*

Definition 2.1.3. *(Special Sets)*

- *The empty set, denoted by $\emptyset$, is a set that does not contain any elements.*
- *A set with only one element is called a singleton.*
- *A set with exactly two distinct elements is called a pair.*
- *The cardinality of a set E, denoted as $card(E)$, is the number of elements in a finite set.*

Definition 2.1.4. *(Set Inclusion) A set F is said to be contained in, a subset of, or included in set E, denoted as $F \subseteq E$, if every element of F is also an element of E. If there exists at least one element in F that is not in E, it is denoted as $F \nsubseteq E$.*

Remark 2.1.5. *(Properties of Set Inclusion)*

- *Every set E is a subset of itself (reflexivity). $E \subseteq E$.*
- *If set F is a subset of set E and set G is a subset of set F, then G is also a subset of E (transitivity). If $F \subseteq E$ and $G \subseteq F$, then $G \subseteq E$.*
- *If E and F are sets such that $E \subseteq F$ and $F \subseteq E$, then E and F have the same elements, and they are equal $E = F$ (Antisymmetry).*
- *If $F \subset E$, then $Card(F) \leq Card(E)$.*

Definition 2.1.6. *(Power set) Let E be a set, the subsets of E form a set called set of parts of E (Power set) and noted $\mathcal{P}(E)$(The set of all subsets of the set E). In other words, $A \in \mathcal{P}(E)$ means $A \subset E$.*

Remark 2.1.7. *Remark that the elements of $\mathcal{P}(E)$ are subsets of E and not elements in E. Moreover, unlike the set E that can be empty, the set of $\mathcal{P}(E)$ can't be empty. since it contains at least E, $\emptyset$.*

Example 2.1.8. *If $E = \{a, b, c\}$ then*

$$\mathcal{P}(E) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, E\}.$$

Proposition 2.1.9. *If E is a finite set of cardinal n, the $\mathcal{P}(E)$ is also finite with $Card(\mathcal{P}(E)) = 2^n$.*

Definition 2.1.10. *(Complement) If $F \subseteq E$, the complement of F in E is the set $C_E F$, also denoted as F^c or F_E^c, defined by*

$$F^c = \{x \in E,\ x \notin F\}.$$

Example 2.1.11. *Let $F = \{x \in \mathbb{N},\ 0 \leq x \leq 6\}$ be a set, it's complement in $\mathbb{N}$ is*

$$F_{\mathbb{N}}^c = \{x \in \mathbb{N},\ x > 6\}.$$

However, it's complement in $\mathbb{Z}$ is

$$F_Z^c = \{x \in \mathbb{Z},\ x < 0 \text{ or } x > 6\} \neq F_{\mathbb{N}}^c.$$

Proposition 2.1.12. *Let F and G two subsets of E, then*

1. $(F^c)^c = F$.
2. $F \cap F^c = \emptyset, \quad F \cup F^c = E$.
3. $F \subseteq G \iff G^c \subseteq F^c$.

Definition 2.1.13. *(Intersection) The intersection of two sets E and F is the set $E \cap F$ consisting of elements x that are both in E and in F. Two sets E and F are said to be disjoint if $E \cap F = \emptyset$.*

$$E \cap F = \{x \mid x \in E \text{ and } x \in F\}.$$

Proposition 2.1.14.

1. $E \cap F = F \cap E$ *(commutativity).*
2. $E \cap (F \cap G) = (E \cap F) \cap G$ *(associativity).*
3. $(E \subset (F \cap G)) \Leftrightarrow [(E \subset F) \text{ and } (E \subset G)]$.

Definition 2.1.15. *(Union) The union of two sets E and F is the set $E \cup F$ consisting of elements x that are in E, or in F, or in both at the same time.*

$$E \cup F = \{x \mid x \in E \text{ or } x \in F\}.$$

Proposition 2.1.16. *Let F, G two subsets of E we have the following relations, called Morgan Laws*

- $(F \cap G)^c = F^c \cup G^c$.
- $(F \cup G)^c = F^c \cap G^c$.

Proposition 2.1.17. *Let F, G two finite parts of a set E, then*

$$Card(F \cup G) = Card(F) + Card(G) - Card(F \cap G).$$

Definition 2.1.18. *(Difference) If E and F are two sets, the difference $E \setminus F$ between E and F is the set of elements of E that are not in F.*

$$E \setminus F = \{x/\ x \in E,\ x \notin F\}$$

The symmetric difference $E \Delta F$ of E and F is given by

$$E \Delta F = (E \setminus F) \cup (F \setminus E).$$

Remark 2.1.19. *If $F \subset G$, then $F \setminus G = \emptyset$.*

Definition 2.1.20. *(Partition of a set)*

1. *For a non-empty set E, a partition $A = \{A_1, A_2, \ldots, A_n\}$ of E is a set of non-empty subsets of E such that $E = \bigcup_{i=1}^{n} A_i$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.*
2. *For a partition $A = \{A_1, A_2, \ldots, A_n\}$ of a set E, each set A_i is called a cell of the partition.*

Definition 2.1.21. *(Cartesian Product) The Cartesian product of two sets E and F is the set*

$$E \times F = \{(x, y) \mid x \in E, y \in F\}.$$

The diagonal of a set E is given by

$$\Delta_E = \{(x, x) \mid x \in E\} \subseteq E \times E.$$

Proposition 2.1.22. *Let E, F two finite non empty sets, we have $Card(E \times F) = Card(E) \times Card(F)$.*