People's Democratic Republic of Algeria

University of science and technology Oran M-B

Department of Mathematics

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Algebra 1

Exercise series N°02

Exercise 1:

1. Let $E = \{a, b, c, d\}$ be a set, can we write the following
   (1) $a \in E$, (2) $a \subset E$, (3) $\{a\} \subset E$, (4) $\emptyset \in E$, (5) $\emptyset \subset E$.
   Give the power set of E.

2. Let $A =]-\infty, 2[$, $B =]-2, 6[$, and $C =]-4, +\infty[$ three sets of $\mathbb{R}$.
   Determine $A \cap B$, $A \cup B$, $B \cap C$, A^c, $A^c \cap B^c$, $A|B$.

Exercise 2:

Let $f : \mathbb{R}_+ \longrightarrow [1, +\infty[$, $g : [1, +\infty[\longrightarrow \mathbb{R}_+$ such that $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-1}$. Is $f \circ g = g \circ f$?

Exercise 3:

Let $g : \mathbb{Z} \longrightarrow \mathbb{N}$ be a function such that $g(n)$ is the number of positif divisors of n.

1. Is g an application?

2. Determine $g(\{-18, 0, 1, 3\})$, $g^{-1}(\{0, 1, 2, 3\})$ and graph $G_g(n \in \mathbb{Z} : |n| \leq 2)$.

3. Determine the restriction of g on P "the set of prime numbers"

Exercise 4:

Are the following functions injective? surjective?

1. $f : \mathbb{N} \longrightarrow \mathbb{N}$, $n \longmapsto n + 1$;

2. $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, $n \longmapsto n + 1$;

3. $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^2 + 1$;

Exercise 5:

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{2x}{1+x^2}$

1. Is f injective? surjective?

2. Prove that $f(\mathbb{R}) = [-1, 1]$.

3. Prove that the restriction $g : [-1, 1] \longrightarrow [-1, 1]$ such that $g(x) = f(x)$ is a bijection.

4. Prove the previous results using the study of variations of f.

Exercise 6:

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2 + x - 2$

1. Define and compute $f^{-1}(4)$.

2. Is f bijective?

3. Define and compute $f([-1,1])$ and $f^{-1}([-2,4])$.

Exercise 7:

Let $f : \mathbb{Z} \longrightarrow \mathbb{Q}$ a function defined by $f(x) = \frac{x}{2}$

1. Determine $f(\mathbb{Z})$, $f^{-1}(B)$ such that $B = \{\frac{5}{3}\}$.

2. Is f injective? surjective?

Exercise 8:

In the following cases, determine $f(I)$, then verify that f realise a bijection from I to $J = f(I)$, then precise f^{-1}.

1. $f(x) = \sqrt{2x+3}$-2, $\quad I =]\frac{-3}{2}, +\infty[$.

2. $f(x) = \frac{1}{1+x^2}$, $\quad I = [0, +\infty[$.

3. $f(x) = \frac{1}{\sqrt{x^2+2x+2}}$, $\quad I = [-1, +\infty[$.

Exercise 9:

Prove that the function

$$\begin{aligned} f :&\mathbb{R} \longrightarrow \mathbb{R}_+^* \\ x &\longmapsto \frac{e^x + 2}{e^{-x}} \end{aligned}$$

is a bijection. Compute its reciprocal bijection f^{-1}.

Exercise 10:

Let $f : E \longrightarrow F$ be an application, and let A and B be two subsets of E, and C, D be two subsets of F. Show that

(a) If $A \subset B$, then $f(A) \subset f(B)$.

(b) $f(A \cup B) = f(A) \cup f(B)$ and $f(A \cap B) \subset f(A) \cap f(B)$.

(c) If $C \subset D$, then $f^{-1}(C) \subset f^{-1}(D)$.

(d) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ and $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$.