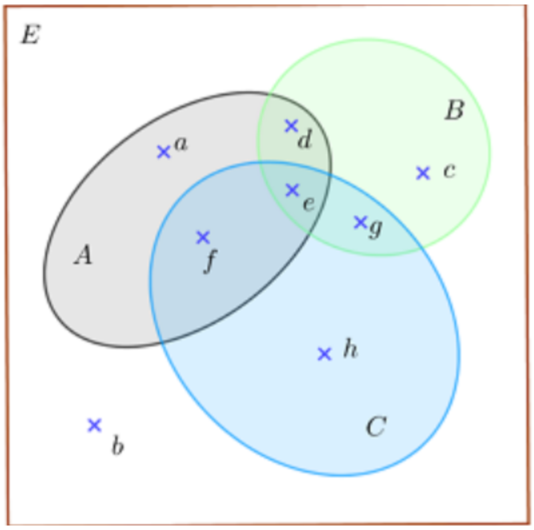
Exercise series N°02

Exercise 1:

Let A, B, C be three parts of a set E, and a, b, c, d, e, f, g, h elements of E. Are these assertions true or false: 1.



$g \in A \cap B$; 2. $g \in A \cup B$; 3. $f \in C \setminus A$; 4. $e \in A \cap B \cap C$; 5. $\{h, b\} \subset A \cap B$; 7. $\{a, f\} \subset A \cup C$.

Exercise 2:

Let E be a set and A, B, C be three elements of $\mathcal{P}(E)$.

1. Prove that, if $A \cap B = A \cup B$, then $A = B$.
2. Prove that, if $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then $B = C$.
3. Does $C \subset A \cup B$ imply $C \subset A$ or $C \subset B$?

Exercise 3:

Let $f : \mathbb{R}_+ \longrightarrow [1, +\infty[$, $g : [1, +\infty[\longrightarrow \mathbb{R}_+$ such that $f(x) = x^2 + 1$ and $g(x) = \sqrt{x - 1}$. Is $f \circ g = g \circ f$?

Exercise 4:

Let the following functions be: $f : \mathbb{N} \to \mathbb{N}$ with $f(n) = n + 2$; $g : \mathbb{Z} \to \mathbb{Z}$ with $g(n) = n + 2$.

1. Determine $f(\{0, 1, 2\})$, $f^{-1}(\{0\})$, $g^{-1}(\{1\})$.
2. Is the function f bijective? Justify.
3. Is the function g bijective? Justify.

Exercise 5:

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$f(x) = x^2 - 1$$

1. Find the direct image $f(\{-1, 1\})$. What can you conclude?
2. Find the inverse image $f^{-1}(\{-2\})$. What can you conclude?
3. Prove that $f([0, \infty)) = [-1, \infty)$.

4. Let $g : [0, \infty) \to [-1, \infty)$ be the function defined by $g(x) = f(x)$. Prove that g is bijective and determine its inverse g^{-1}.

Exercise 6:

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$f(x) = \frac{x^2}{1 + x^2}$$

1. Find the direct image $f(A)$ of $A = \{-1, 0, 1\}$ under f.
2. Find the inverse image $f^{-1}(B)$ of $B = \{-1\}$ under f.
3. Is f injective? Surjective? Justify your answer.
4. Consider the function $g : (-\infty, 0] \to [0, 1]$ defined by:

$$g(x) = f(x), \quad \forall x \in (-\infty, 0].$$

   Show that g is bijective and define its inverse g^{-1}.

Exercise 7:

In the following cases, determine $f(I)$, then verify that f realise a bijection from I to $J = f(I)$, then precise f^{-1}.

1. $f(x) = \sqrt{2x+1}$-1, $\quad I =]\frac{-1}{2}, +\infty[$.
2. $f(x) = \frac{1}{1+x^2}, \quad I = [0, +\infty[$.
3. $f(x) = \frac{1}{\sqrt{x^2+2x+2}}, \quad I = [-1, +\infty[$.

Exercise 8:

Prove that the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}_+^*$$
$$x \longmapsto \frac{e^x + 2}{e^{-x}}$$

is a bijection. Compute its reciprocal bijection f^{-1}.

Exercise 9:

Let f from E to E be an application such that:

$$f(f(E)) = E$$

Show that f is onto.