2.2 Functions and Applications

2.2.1 Functions

Definition 2.2.1. *A correspondence f from E to F is called a function if every element x in E has at most one image y in F.*

- *We say that E is the domain or (the source set), and F is called the codomain or (the target set).*
- *The element associated to x by f, is called the image of x and it is noted $f(x)$ (means $y = f(x)$).*
- *The domain of definition of a function f (denoted by D_f) is the set of elements x of E for which $f(x)$ exists.*

Examples 2.2.2. *1. The correspondence f that associates each natural number with the corresponding month is a function from $\mathbb{N}$ to the set*
$B = \{January, February, March, April, May, June, July, August, September, October, November, December\}$.
In this case, $f(2) = february$, and $f(15)$ does not exist.
$D_f = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

2. *The correspondence that associates each month with the possible number of days in the month is not a function from the set B in the previous example to $\mathbb{N}$, because it associates two elements, 28 and 29, to February.*

3. *The correspondence g that associates each integer with its square is indeed a function, and we can write it as $g : \mathbb{N} \longrightarrow \mathbb{N}$, for $g(n) = 2n$, the domain of g is $D_g = \mathbb{N}$.*

Definition 2.2.3. *Let*

$$\begin{aligned} f :& E \longrightarrow F \\ & x \longmapsto f(x) \end{aligned}$$

be a function, where A is a subset of E and B is a subset of F.

1. *The image of A by f is*

$$f(A) = \{f(x),\ x \in A \cap D_f\}.$$

2. *The preimage (or inverse image) of B by f is*

$$f^{-1}(B) = \{x \in E,\ f(x) \in B\}.$$

3. *Let $f : E \longrightarrow F$, if $A \subset E$, we call graph of A, and we note it $G_f(A)$, the subset of $E \times F$ formed by the couples $(x, f(x))$ such that $x \in A \cap D_f$. Which means*

$$G_f(A) = \{(x, f(x)) \in E \times F / x \in A \cap D_f\}.$$

Examples 2.2.4. 1. *If we take the function given in the previous Example (2.2.2), we will have:* $G_f(\{1, 4\}) = \{(1, January), (4, April)\}$
$f(2\mathbb{N}) = \{August, October, February, April, June, December\}$
$D_f = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, *and* $f^{-1}(\{June, December\}) = \{6, 12\}$.

2. *Let $g : \mathbb{N} \longrightarrow \mathbb{N}$ be a function such that $g(n) = n^2$.* $G_g = \{(n, n^2)/n \in \mathbb{Z}\}$
$g(\{-1, 1, 0, 2, 3\}) = \{0, 1, 4, 9\}$, $g^{-1}(\{9\}) = \{-3, 3\}$.

3. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = \frac{1}{x}$.*
$G_h = \{(x, \frac{1}{x})/x \in \mathbb{R}^*\}$, $h([-1, 2]) =]-\infty, -1] \cup [\frac{1}{2}, +\infty[$, *and*

$$h^{-1}([2, 3]) = [\frac{1}{3}, \frac{1}{2}].$$

2.2.2 Representations of Functions

The representation of a function $f : E \longrightarrow F$ depends on the nature of the sets E and F. The most commonly used representations are as follows

1. **Representation using a formula**.

   Example: Let's consider the function $g : \mathbb{Z} \longrightarrow \mathbb{N}$ such that $g(n) = n^2$.

2. **Representation using a table of values** (useful when A is finite).

   Example: Let's consider the function $h : \{-2, -1, 0, 1, 2\} \longrightarrow \mathbb{N}$ such that:

n	-2	-1	0	1	2
h(n)	4	1	3	1	0

3. **Representation using a graph**. Representation using a formula. Example: Let's consider the function $k : \mathbb{R} \longrightarrow \mathbb{R}$ such that $k(x) = x^2$.

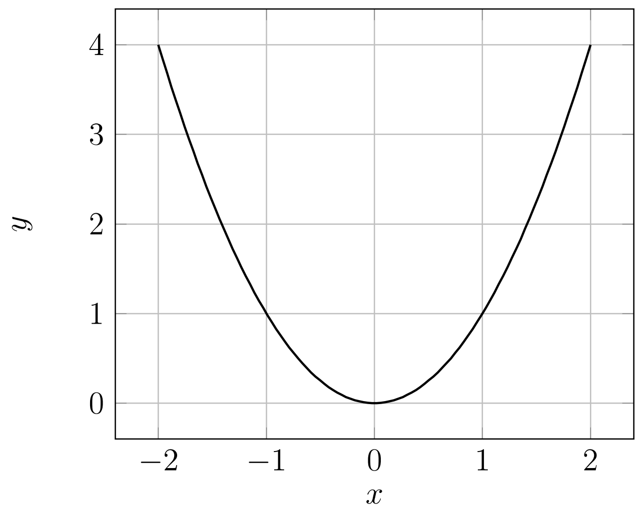

Figure 2.1: The graph of $k(x) = x^2$.

Definition 2.2.5. *(Composition of functions) The composition of the function $f : E \longrightarrow F$ and the function $g : F \longrightarrow G$ is the function*

$$\begin{aligned} g \circ f : & E \longrightarrow G \\ & x \longmapsto g(f(x)). \end{aligned}$$

Example 2.2.6. *Let the functions f, g defined from $\mathbb{R}$ to $\mathbb{R}$ given by $f(x) = 3x - 2$ and $g(x) = x^2$. The composition of f followed by g is the function $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$, such that $g \circ f(x) = g(f(x)) = (3x - 2)^2$.*

2.2.3 Applications

Definition 2.2.7. *A function f is an application if every element of E has (exactly) one image in F. We denote by $\mathcal{F}(E, F)$ the set of all applications from E to F. A function f is an application if and only if its domain of definition is all of E.*

Examples 2.2.8.

1. *The function $g : \mathbb{Z} \longrightarrow \mathbb{N}$, defined by $g(n) = n^2$, is a mapping from $\mathbb{Z}$ to $\mathbb{N}$.*

2. *The function $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = \frac{1}{x}$ is not a mapping (application) because $D_f = \mathbb{R}^* \neq \mathbb{R}$.*

3. *The function $Id_E : E \longrightarrow E$, defined by $Id_E(x) = x$, is a specific mapping called the identity mapping of E.*

2.2.4 Restriction and Extension

Let $f : E \longrightarrow F$ be a mapping.

1. The restriction of f to a subset E_0 of E is the mapping $g : E_0 \longrightarrow F$ defined by $g(x) = f(x)$ for all $x \in E_0$ (g is often denoted as $f_{|E_0}$).

2. The extension of f to a set $\tilde{E}$ containing E is the function $h : \tilde{E} \longrightarrow F$ defined by $h(x) = f(x)$ for all $x \in E$.

Example 2.2.9. *Let the mapping $f : \mathbb{Z} \longrightarrow \mathbb{N}$ be defined by $f(n) = |n|$. The restriction of f to $\mathbb{N}$ is the identity mapping $Id_{\mathbb{N}}$. We can also say that the mapping f is an extension of $Id_{\mathbb{N}}$.*

Remark 2.2.10. *The restriction is always unique, but an extension is not unique.*

2.2.5 Equality of mappings

Two mappings $f : E \longrightarrow F$ and $g : E' \longrightarrow F'$ are equal if $E = E'$, $F = F'$, and for all $x \in E$, we have $f(x) = g(x)$. In this case, we write $f = g$.

Example 2.2.11. *The mappings f and g defined from $\mathbb{N}$ to $\mathbb{Z}$ by $f(n) = cos(\pi n)$ and $g(n) = (-1)^n$ are equal, and we can write $f = g$.*

Proposition 2.2.12. *Let $f : E \longrightarrow F$ be an application.*

1. *Let A and B be two subsets of F. Then*
   - **(a)** *If $A \subset B$, then $f^{-1}(A) \subset f^{-1}(B)$.*
   - **(b)** *We always have $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.*
   - **(c)** *We always have $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.*
2. *Let A and B be two subsets of E. Then*
   - **(a)** *If $A \subset B$, then $f(A) \subset f(B)$.*
   - **(b)** *We always have $f(A \cup B) = f(A) \cup f(B)$.*
   - **(c)** *We always have $f(A \cap B) \subset f(A) \cap f(B)$.*
3. **(a)** *If A is a subset of E, then $A \subseteq f^{-1}(f(A))$.*
   - **(b)** *If B is a subset of F, then $f(f^1(B)) \subseteq B$.*