2.2.6 Injection, Surjection, Bijection

Definition 2.2.13. *(injection"one to one") Let E and F be two sets, and let $f : E \longrightarrow F$ be a function.*
f is injective if every element of F has at most one pre-image in E. In other words:

$$\forall x, y \in E,\ f(x) = f(y) \Rightarrow x = y.$$

Also, it can be written using the contrapositive as follow

$$\forall x, y \in E,\ x \neq y \Rightarrow f(x) \neq f(y).$$

Examples 2.2.14.

1. *The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = 3x + 1$ is one to one, since*
   $f(x) = f(y)$ implies $3x + 1 = 3y + 1$. Hence $x = y$.
2. *The function $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ such that $f(x) = \frac{1}{x}$ is one to one, since*
   $f(x) = f(y)$ implies $\frac{1}{x} = \frac{1}{y}$. Hence $x = y$.
3. *The function $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ such that $f(x) = x^2$ is not one to one, since $f(-1) = f(1)$.*

Theorem 2.2.15. *Let $f : E \longrightarrow F$ be a function. The following assertions are equivalents*

1. *f is injective.*
2. *For all $(x, y) \in E^2$, $x \neq y$ implies $f(x) \neq f(y)$.*
3. *For all $b \in F$, the equation $f(x) = b$ has at most one solution x.*

Proof : To prove this theorem, it is sufficient to prove 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1).

- 1) $\Rightarrow$ 2), this implication can be obtained easily by using the contrapositive of the injectivity definition.
- 2) $\Rightarrow$ 3), suppose that the equation $f(x) = b$ has two solutions x, y or more, which means $x \neq y$, using 2), we get $f(x) \neq f(y)$,i.e., $b \neq b$ which is a contradiction. Hence, the equation $f(x) = b$ has at most one solution.

- 3) $\Rightarrow$ 1), Let $x, y \in E$ such that $f(x) = f(y)$, then x is a solution of $f(x) = b$, $b \in F$, and also y is a solution of $f(y) = b$. Using 3), x can't be different of y, which means $x = y$. Hence, f is injective.

□

Definition 2.2.16. *(Surjection"Onto") f is surjective if every element of F has at least one pre-image in E. In other words:*

$$\forall y \in F,\ \exists x \in E, f(x) = y.$$

Examples 2.2.17. *1. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = 3x + 1$ is surjective, since*

$$\forall y \in \mathbb{R},\ \exists x = \frac{y-1}{3} \in \mathbb{R}, f(x) = y.$$

2. The function $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ such that $f(x) = \frac{1}{x}$ is not surjective, since $y = 0$ has no antecedent.

3. The function $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ such that $f(x) = x^2$ is surjective, since

$$\forall y \in \mathbb{R}^+,\ \exists x = \pm\sqrt{y} \in \mathbb{R}, f(x) = y.$$

Theorem 2.2.18. *Let $f : E \longrightarrow F$ be a function. The following assertions are equivalents*

1. *f is surjective.*
2. *$f(E) = F$.*
3. *For all $b \in F$, the equation $f(x) = b$ has at least one solution x.*

Proof :

- Show that 1) implies 2). If f is surjective, then for every y in F, there exists x in E such that $y = f(x)$. Thus, y is in $f(E)$, and since $f(E)$ is a subset of F, it follows that 2) holds.

- Show that 2) implies 3). If 2) holds, then for every $b \in F$, there exists at least one x in E such that $b = f(x)$, which means x is a solution to the equation.

- Show that 3) implies 1). If 3) holds, then for every y in F, there is at least one solution x to the equation $f(x) = y$, which means x is a pre-image of y.

□

Definition 2.2.19. *(Bijection) f is bijective if it is both injective and surjective (every element of F has exactly one pre-image in E).*

Examples 2.2.20. *1. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = 3x + 1$ is bijective, since it is injective and surjective.*

2. The function $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ such that $f(x) = \frac{1}{x}$ is not bijective, since it is not surjective.

3. The function $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ such that $f(x) = x^2$ is not bijective, since it is not injective.

Theorem 2.2.21. *Let $f : E \longrightarrow F$ be a function. The following assertions are equivalents*

1. f is bijective.

2. For all $b \in F$, the equation $f(x) = b$ has a unique solution x.

Proof : A function is bijective if and only if the equation $y = f(x)$ has at least (see Theorem 2.2.15) and at most (see Theorem 2.2.18) one solution, hence a unique solution.

□

2.2.6.1 Reciprocal application of a bijective function

Definition 2.2.22. *Let $f : E \longrightarrow F$ be a bijective function. The reciprocal application of f, denoted by f^{-1}, is defined as $f^{-1} : F \longrightarrow E$, where $f^{-1}(y) = x$, and x is the antecedent of y by f (i.e., $f(x) = y$).*

Example 2.2.23. *The bijection f defined from $\mathbb{R}$ to $\mathbb{R}$ by $f(x) = 3x + 1$, its reciprocal application is defined from $\mathbb{R}$ to $\mathbb{R}$ by $f^{-1}(x) = \frac{x-1}{3}$.*

Theorem 2.2.24. *Let $f : E \longrightarrow F$ be a bijective function. Then*

(a) *The reciprocal application f^{-1} is bijective and $(f^{-1})^{-1} = f$.*

(b) *$f \circ f^{-1} = Id_F$ and $f^{-1} \circ f = Id_E$.*

Proof :

(a) For each $x \in E$, the equation $f^{-1}(y) = x$ has a unique solution $y = f(x)$ and it is unique because another solution y' can only be $f(x)$. Then, according to Theorem 2.2.21, f^{-1} is bijective. Moreover, $(f^{-1})^{-1} : E \longrightarrow F$ and $(f^{-1})^{-1}(x) = y$ since $f^{-1}(y) = x$. Therefore, $(f^{-1})^{-1} = f$.

(b) We have $f : E \longrightarrow F$ and $f^{-1} : F \longrightarrow E$, then $f \circ f^{-1} : F \longrightarrow F$. Also, $f \circ f^{-1}(y) = f(x) = y = Id_F(y)$, hence the equality $f \circ f^{-1} = Id_F$. Similarly, it can be shown that $f^{-1} \circ f = Id_E$.

□

Theorem 2.2.25. *(Bijection Theorem) Let I be an interval in $\mathbb{R}$. Let $f : I \to \mathbb{R}$. We assume that f is continuous and strictly monotonic on I. Then*

- *f establishes a bijection from I to the interval $J = f(I)$.*
- *f^{-1} is monotonically increasing on J, with the same direction of variation as f, and $f^{-1} : J \to I$ with $f^{-1}(y) = x$ (x is the antecedent of y by f).*