Chapter 1

Logic Concepts

In this chapter, we will introduce the fundamental elements of classical logic.

1.1 Assertions (Propositions)

Definition 1.1.1. *An assertion is a sentence that is either true or false, but not both at the same time.*

Example 1.1.2.

1. *"$\sqrt{2}$ is an irrational number" is a true assertion.*
2. *"16 is a multiple of 2" is a true assertion.*
3. *"19 is a multiple of 2" is a false assertion.*
4. *"$9 \geq 8$" is a true proposition.*
5. *"Good moorning" is not a logical statement.*

Assertions are represented by uppercase letters $P, Q, R, \ldots$ If an assertion is true, we assign it the value 1 (or T); if it is false, we assign it the logical value 0 (or F).

Truth Table for Assertion P

P
1
0

Logical Connectors

Logical connectors allow us to create compound statements, called composite assertions, from assertions P, Q, R, ... We can determine the truth value of these compound assertions based on the truth values of P, Q, R, ... The most common logical connectors are “not,” “and,” “or,” “implies,” and “if and only if.”

Negation

The negation of the assertion P is denoted as $\neg P$ (or sometimes $\bar{P}$), and it is true when P is false and false when P is true. For the logical connector not, we have the following truth table:

P	$\neg P$
0	1
1	0

Example 1.1.3. *"12 is a multiple of 2" is a true statement. Its negation is "12 is not a multiple of 2," which is a false statement. "14 is a multiple of 3" is a false statement. Its negation is "14 is not a multiple of 3," which is a true statement. In this example:*

It is important to note that the truth value of an assertion and its negation can be evaluated independently. In this case, we can see that the original statement can be true while its negation is false, and vice versa.

Conjunction

The conjunction logical connector is represented by the symbol $\wedge$. It allows us to combine two propositions into a single proposition that is true only if both original propositions are true. For example, if P and Q are true propositions, then $P \wedge Q$ (P and Q) is also true.

Let P, Q two logical statement, we have the following truth table:

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction

The disjunction logical connector is represented by the symbol $\vee$. It allows us to combine two propositions into a single proposition that is true if at least one of the original propositions is true. For example, if P is a true proposition and Q is a false proposition, then $P \vee Q$ (P or Q) is true.
Let P, Q two logical statement, we have the following truth table:

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

Remark 1.1.4. *In general, the proposition $P \wedge Q$ is true if both P and Q are true, and false otherwise. The proposition $P \vee Q$ is true if at least one of the propositions P and Q is true, and false otherwise.*

Note that if both P and Q are true, then $P \vee Q$ is true. This is called an "inclusive or."

Remark 1.1.5. *1. For any proposition P, one of the two propositions P or $\neg P$ is true, and the other is false. It follows that the proposition $P \wedge \neg P$ is always false, and the proposition $P \vee \neg P$ is always true.*

2. *The negation of $P \wedge Q$ is $\neg P \vee \neg Q$, and the negation of $P \vee Q$ is $\neg P \wedge \neg Q$*

3. *The proposition "$Q \vee P$" means the same thing as the proposition "$P \vee Q$". Similarly, the proposition "$Q \wedge P$" means the same thing as the proposition "$P \wedge Q$". We say that $\wedge$ and $\vee$ are commutative.*

4. *We can combine $\wedge$, $\vee$, and $\neg$ to form new propositions. It is important to pay attention to the placement of parentheses because the meaning depends on it. For example, let's assume that proposition P is false and proposition Q is true. In this case, $P \wedge (\neg P \vee Q)$ is false (because P is false, so P AND (any proposition) is always false, regardless of that (proposition)). However, $(P \wedge \neg P) \vee Q$ is true (because Q is true, so (any proposition) OR Q is true, regardless of that (proposition)).*

Implication

The implication logical connector is represented by the symbol $\Rightarrow$ (or $\rightarrow$). It establishes a relationship between two propositions, where the first proposition (called the antecedent) implies the second proposition (called the consequent). If the antecedent is true, then the consequent must be true, but if the antecedent is false, we cannot draw any conclusions about the consequent.

Let P, Q two logical statement, we have the following truth table:

P	Q	$P \Rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

Remark 1.1.6. 1. *In practice, if P, Q, and R are three assertions, then the composite assertion $(P \rightarrow Q) \wedge (Q \rightarrow R)$ is written as $(P \rightarrow Q \rightarrow R)$.*

2. *The implication $Q \rightarrow P$ is called the reciprocal implication of $P \rightarrow Q$.*

3. *The implication $\neg Q \rightarrow \neg P$ is called the contrapositive implication of $P \rightarrow Q$.*

Equivalence

An assertion called the equivalence of P and Q and written as $P \Leftrightarrow Q$, is a statement that is true when P and Q are both true or both false, and false in all other cases. In other words, if P is true, then Q is true, and if P is false, then Q is false.

Vocabulary: To say that P is equivalent to Q, we also say that P is true if and only if Q is true. We also say that P is a necessary and sufficient condition for Q.

The table below shows that proposition P is equivalent to proposition Q if and only if they are both true or both false.

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$P \leftrightarrow Q$
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

Properties of Equivalence:

1. P is equivalent to P (equivalence is reflexive).

2. If P is equivalent to Q, then Q is equivalent to P (equivalence is symmetric).

3. If P is equivalent to Q and Q is equivalent to R, then P is equivalent to R (equivalence is transitive).

4. P is equivalent to Q if and only if "not P" is equivalent to "not Q."

Example of Transitivity of Equivalence:

Let x and y be real numbers. Let P, Q, and R be the following propositions:

$$P : x + 1 > y + 1$$

$$Q : x > y$$

$$R : x - y > 0$$

Since P is equivalent to Q and Q is equivalent to R, we can conclude that P is equivalent to R.

1.2 Mathematical Quantifiers

Definition 1.2.1. *Let E be a set. A predicate on E is a statement containing variables, such that when each of these variables is replaced by an element of E, we obtain a proposition (statement). A predicate containing the variable x will be denoted as $P(x)$.*

Example 1.2.2. *The statement $P(n)$ defined as "n is a multiple of 2" is a predicate on $\mathbb{N}$. It becomes a proposition when we assign an integer value to n. For example:*

1. *The proposition $P(10)$ defined as "10 is a multiple of 2" is true when n is replaced by 10.*

2. *The proposition $P(11)$ defined as "11 is a multiple of 2" is false when n is replaced by 11.*

Starting from a predicate $P(\cdot)$ defined on a set E, we can construct new statements called quantified statements using the quantifiers "there exists ($\exists$)" and "for all ($\forall$)".

Definition 1.2.3. *Let $P(\cdot)$ be a predicate defined on a set E.*

1. *The quantifier "for all" (also called "for every") denoted as $\forall$, allows defining the quantified statement "$\forall x \in E, P(\cdot)$" which is true when all elements x of E satisfy $P(\cdot)$.*

2. *The quantifier "there exists" denoted as* $\exists$*, allows defining the quantified statement "*$\exists x \in E, P(x)$*" which is true when at least one element* x_0 *belonging to* E *satisfies the statement* $P(x)$*.*

Remark 1.2.4. *There exists a unique* $x \in E$*,* $P(x)$*, is noted "*$\exists! x \in E, P(x)$*".*

Example 1.2.5. 1. *The statement "*$x^2 + 2x - 3 < 0$*" is a predicate defined on* $\mathbb{R}$*. It can be true or false depending on the value of* x*. The quantified statement "*$\forall x \in]-3, 1[, x^2 + 2x - 3 < 0$*" is a true statement because the quantity* $x + 2x - 3$ *is negative or zero for all* x *belonging to the closed interval* $]-3, 1[$*.*

2. *The quantified statement "*$\exists x \in \mathbb{R}, x^2 = 4$*" is true because there exists (at least) one element* x *in* $\mathbb{R}$ *that satisfies* $x^2 = 4$*. This is the case for the two real numbers -2 and 2.*

Negation Rules for Quantified Statements

The negation of "for every element x in E, the statement $P(x)$ is true" is "there exists an element x in E for which the statement $P(x)$ is false."

$$\overline{(\forall x \in E,\ P(x))} \Leftrightarrow (\exists x \in E, \overline{P(x)}).$$

The negation of "there exists an element x in E for which the statement $P(x)$ is true" is "for every element x in E, the statement $P(x)$ is false."

$$\overline{(\exists x \in E,\ P(x))} \Leftrightarrow (\forall x \in E, \overline{P(x)}).$$

Example 1.2.6. 1. *The negation of* $(\exists x \in (0, +\infty), x > 1)$ *is* $(\forall x \in (0, +\infty), x \leq 1)$*.*

2. *The negation of* $(\forall x \in \mathbb{R}, 2x + 2 = 0)$ *is* $(\exists x \in \mathbb{R}, 2x + 2 \neq 0)$*.*

Remark 1.2.7. *It is not more difficult to write the negation of complex sentences. For the assertion:*

$(\forall x \in E, p(x) \Rightarrow Q(x))$*, its negation is* $(\exists x \in E, p(x) \wedge \overline{Q(x)})$