1.3 Logical reasoning for proofs

There are many ways to prove mathematical statements. Here we will outline some usual techniques.

Direct reasoning

To show that the logical implication $P \Rightarrow Q$ is true, we need to assume that P is true and demonstrate that Q is true as a result. Which means, suppose that P is true (assumption). Use logical laws and reasoning to deduce Q from P.

Example 1.3.1. *Prove that, if $a, b \in \mathbb{Q}$, $a + b \in \mathbb{Q}$.*
We have a, $b \in \mathbb{Q}$, means $a = \frac{p}{q}$ and $b = \frac{p'}{q'}$, with $p, p' \in \mathbb{Z}$ and $q, q' \in \mathbb{Z}^*$ then

$$a + b = \frac{pq' + p'q}{qq'} \in \mathbb{Q}, \ (pq' + p'q \in \mathbb{Z} \text{ and } qq' \in \mathbb{Z}^*).$$

Case by case reasoning

To verify an assertion $P(x)$ for all $x \in E$, we prove the assertion for x belonging to a set A of E, then for the x that doesn't belong to A. This is the disjunction method or case by case.

Example 1.3.2. *Prove that $\forall n \in \mathbb{N}$, $\frac{n(n+1)}{2}$ is an integer.*
We distinct two cases

(a) *If n is even, then $n = 2k$, and $n + 1 = 2k + 1$ with $k \in \mathbb{N}$, which means*

$$\frac{n(n+1)}{2} = k(2k+1),$$

which is clearly an integer number.

(b) *If n is odd, then $n = 2k+1$, and $n+1 = 2k+2 = 2(k+1)$ with $k \in \mathbb{N}$, which means*

$$\frac{n(n+1)}{2} = (k+1)(2k+1),$$

which is also an integer number.

Contrapositive reasoning

Contrapositive reasoning is a logical method based on the principle that if a logical implication is true, then its contrapositive is also true. Consider a logical implication of the form "If P, then Q" (P implies Q). The contrapositive of this implication is "If not Q, then not P" ($\overline{Q}$ implies$\overline{P}$). To demonstrate the validity of the initial implication, we can prove that its contrapositive is true. Contrapositive reasoning is useful when a direct proof of the initial implication is difficult. By proving the contrapositive, we can obtain a simpler or more direct proof.

Example 1.3.3. *Let $a, b > 0$. Prove that, if $a \neq b$ then $\frac{a}{b+1} \neq \frac{b}{a+1}$.*
The contrapositive of this proposition is "if $\frac{a}{b+1} = \frac{b}{a+1}$ then $a = b$"

$$\frac{a}{b+1} = \frac{b}{a+1} \Rightarrow a(a+1) = b(b+1)$$
$$\Rightarrow (a-b)(a+b+1) = 0.$$

This implies $a - b = 0$ or $a + b + 1 = 0$. Since $a,\ b > 0$, then $a = b$.

Contradiction reasoning

Proof by contradiction is a logical method used to prove a proposition by initially assuming its negation and then demonstrating that it leads to a contradiction or an absurdity.

Example 1.3.4. *proof that $\sqrt{2}$ is not a rational number.*
Suppose that $\sqrt{2} = \frac{m}{n}$, with pgcd$(m, n) = 1$, so that $2n^2 = m^2$. Which means that m^2 is even, this implies that m is even. which means $m = 2k$, $k \in \mathbb{Z}$. Then $2n^2 = 4k^2$, which implies that $n^2 = 2k^2$. Hence, m^2 is even, this implies that n is even. Therefore, 2 is a comon divisors for n and m, which is a contradiction with pgcd$(m, n) = 1$.

Reasoning by counter-example

The reasoning by counterexample is a logical method used to refute a proposition or statement by presenting a concrete example that contradicts it. If we need to prove an assertion of the form $\forall x \in E,\ P(x)$ is true, we need to prove $P(x)$ true for all $x \in E$. However, if we want to prove that this assertion is false, then we need to find $x \in E$ such that $P(x)$ is false.

Example 1.3.5. *Prove that the following assertion is false "Every positif integer is the sum of three squares."*
Let $a = 7$ an integer number. However, the squares less than 7 are just 1 and 4, which means 7 can't be the sum of three squares. Hence, the propositions is false.

Induction reasoning

Induction is a method of mathematical proof used to establish that a statement $P(n)$ holds true for all natural numbers.
The process of mathematical induction consists of three main steps:

1. We prove that the statement is true for the first value, n_0. This serves as the base case for the induction.

2. We assume that the statement is true for an arbitrary value k, which is known as the inductive hypothesis.

3. We prove that if the statement is true for k, it is also true for $k+1$. This completes the induction step.

Example 1.3.6. *Prove that* $1+2+3+...+n = \frac{1}{2}n(n+1)$.

Note the proposition $1+2+3+...+n = \frac{1}{2}n(n+1)$ by (P_n). Let $n = 1$, P_1 is true $(1 = \frac{1}{2}(1)(2))$.
Suppose that (P_n) is true and prove that (P_{n+1}) is true.

$$(P_{n+1}):\ 1+2+3+...+n+(n+1) = \frac{1}{2}(n+1)(n+2).$$

We have

$$\begin{aligned}1+2+3+...+n+(n+1) &= \frac{1}{2}n(n+1)+(n+1)\\ &= (n+1)(\frac{1}{2}n+1)\\ &= \frac{1}{2}(n+1)(n+2).\end{aligned}$$

Which means (P_{n+1}) is true. Then the proposition (P_n) is true.