People's Democratic Republic of Algeria

University of science and technology Oran M-B — Academic year 2023/2024

Department of Mathematics — Algebra 1

Exercise series N°01

Exercise 1:

Let P, Q and R three propositions, prove that

1. $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$; $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$.
2. $P \Rightarrow Q \equiv \overline{P} \vee Q$; $\overline{P \Rightarrow Q} \equiv P \wedge \overline{Q}$; $P \Rightarrow Q \equiv \overline{Q} \Rightarrow \overline{P}$.

Exercise 2:

1. Let A, B two sets in $\mathbb{R}$, write using $\forall$, $\exists$ the following assertions: $A = \emptyset$, $A \cap B \neq \emptyset$, $A \subset B$, $A \not\subset B$.
2. Give the negation of the following assertions, Are these assertions true or false?

   a) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x + y > 0.$

   b) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x + y > 0.$

   c) $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ xy \geq 0.$

Exercise 3: Let $f : \mathbb{R} \to \mathbb{R}$ is a function, give the contrapositive of the following assertions

1. $\forall x \in \mathbb{R},\ f(x) > 0 \Rightarrow x \leq 0.$
2. $\forall \epsilon > 0,\ \exists \eta > 0,\ \forall (x, y) \in \mathbb{R}^2,\ (|x - y| \leq \eta \Rightarrow |f(x) - f(y)| \leq \epsilon).$
3. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, f(x) = f(y) \Rightarrow x = y.$

Exercise 4:

Using the Contrapositive reasoning, show that

1. Let n be a positive integer, if n^2 is odd, then n is odd.
2. $\forall n \in \mathbb{N}$, n is a prime number $\Rightarrow n = 2$ or n is odd.

Exercise 5:

By the Contradiction reasoning, prove that

1. $\sqrt{2} + \sqrt{3}$ is irrational.
2. If n is the square of an integer, then $2n$ is not a square of an integer.

Exercise 6:

Using a counter example, show that the following propositions are false

1. $\forall a, b \in \mathbb{R}, (a + b)^2 = a^2 + b^2.$
2. $\forall n \in \mathbb{N}, 3n - 4 \in \mathbb{N}.$
3. Every even function is positive.

Exercise 7:

Using disjunction reasoning(case by case), prove that

1. $\forall x \in \mathbb{R}, |x - 1| \leq x^2 - x + 1.$
2. $\forall n \in \mathbb{N},\ \frac{n(n+1)}{2}$ is an integer.

Exercise 8:

Prove by recurrence that

1. $\forall n \in \mathbb{N},\ 4^n + 6n - 1$ is multiple of 9.
2. $\forall n \in \mathbb{N}^*,\ \sum_{k=1}^{n} k(k+1)(k+2) = \frac{1}{4} n(n+1)(n+2)(n+3).$