Chapter 3

Relations

3.1 Generalities of relations

Definition 3.1.1. *A relation from a set A to a set B is any correspondence $\mathcal{R}$ that links elements of A to elements of B in a certain way.*

1. *We call A the domain and B the codomain of relation R.*

2. *If x is linked to y by relation $\mathcal{R}$, we say that x is in relation $\mathcal{R}$ with y, and we write $x\mathcal{R}y$ or $\mathcal{R}(x,y)$. Otherwise, we write $x \not\mathcal{R}(y)$ or $\not\mathcal{R}(x,y)$.*

3. *A relation from A to A is called a relation on A.*

Examples 3.1.2.
1. *Let A be the set of university professors in Usto, and B the set of students in Usto university. We can determine a relation $\mathcal{R}$ from A to B by defining that $(x,y) \in A \times B$ satisfies $x\mathcal{R}y$ if and only if x teaches y.*

2. *Let $A = B = \mathbb{Z}$. We can determine a relation $\mathcal{R}$ on $\mathbb{Z}$ by defining that $(x,y) \in \mathbb{Z}\times\mathbb{Z}$ satisfies $\mathcal{R}(x,y)$ if and only if $x-y$ is even. We have, for example, $1\mathcal{R}9$ but $14 \not\mathcal{R}3$.*

Definition 3.1.3. *(Graph of a relation) The graph of $\mathcal{R}$ (denoted $G_{\mathcal{R}}$) is the set defined by*

$$G_{\mathcal{R}} = \{(x,y) \in A \times B \mid x\mathcal{R}y\}.$$

For example, considering the relation R from the previous example, we have $(1, 7) \in G_{\mathcal{R}}$ and $(18, 7) \notin G_{\mathcal{R}}$.

Example 3.1.4. *Given two relations $\mathcal{R} = (A, B, G_{\mathcal{R}})$ and $\mathcal{R}' = (A', B', G_{\mathcal{R}'})$, the statement "the relations $\mathcal{R}$ and $\mathcal{R}'$ are equal" means that $A = A'$, $B = B'$, and $G_{\mathcal{R}} = G_{\mathcal{R}'}$ same source, same target, and same graph.*

3.2 Representation of a binary relation

We are once again interested in binary relations on two given sets A and B.

1. Set representation: Simply list the pairs satisfying the relation.

2. Representation using a sagittal diagram: A diagram with two curves for arbitrary A and B (one curve for the source A and the other for the target B). When $A = B$, you can either keep the two-curve representation or bring everything together in a single curve representing A. This latter view is often very instructive.

3. Representation using a formula: For example, the relation $\mathcal{R}$ on $\mathbb{R}$ such that $x\mathcal{R}y$ if and only if $x^2 = y^2$.

3.3 Properties of a Binary Relation on a Set

We now focus on a binary relation where the source coincides with the target. Thus, we have a relation on a given set A. Here, we explore the main properties that such a binary relation may or may not possess.

Definition 3.3.1. *Let $\mathcal{R}$ be a (binary) relation on a set A. We say that $\mathcal{R}$ is:*

1. *Reflexive, for every $a \in A$, we have $a\mathcal{R}a$.*

2. *Symmetric, for every pair $(a, b) \in A^2$, if $a\mathcal{R}b$, then $b\mathcal{R}a$.*

3. *Transitive, for every triplet of elements $a, b, c \in A$, if ($a\mathcal{R}b$ and $b\mathcal{R}c$), then ($a\mathcal{R}c$).*

4. Antisymmetric, for every $(a,b) \in A^2$, if $(a\mathcal{R}b$ and $b\mathcal{R}a)$, then $(a=b)$.

Examples 3.3.2. *Let $A = B = \mathbb{Z}$, and $\mathcal{R} = (a,b) \in \mathbb{Z}^2,\ 2|(a-b)$. Then $\mathcal{R}$ is reflexive, symmetric, transitive, but not antisymmetric.*
Given the set $\mathcal{U}$, the inclusion relation, which relates subsets of $\mathcal{U}$ as follows $(X \subseteq Y)$, is reflexive, transitive, and antisymmetric, but not symmetric.
Let the relation $\mathcal{R}$ be defined on $\mathbb{Z}$ as follows: $a\mathcal{R}y \Leftrightarrow a$ divides y.

(a) *For any $a \in \mathbb{Z}$, we have a divides a. So, for all $a \in \mathbb{Z}$, $a\mathcal{R}a$ holds, which means $\mathcal{R}$ reflexive.*

(b) *For $x, y \in \mathbb{Z}$, $x\mathcal{R}y$ implies (x divides y) $\nRightarrow$ (y divides x). For example, 1 divides 4, but 4 does not divide 1, hence $\mathcal{R}$ is not symmetric.*

(c) *Let $x, y \in \mathbb{Z}$, we have $(x\mathcal{R}y)$ and $(y\mathcal{R}x) \Rightarrow$ ((x divides y) and (y divides x)) $\nRightarrow$ $x = y$, for example, (1 divides -1) and (-1 divides 1), but $-1 \neq 1$, hence $\mathcal{R}$ is not antisymmetric.*

(d) *For $x, y, z \in \mathbb{Z}$, if $(x\mathcal{R}y)$ and $(y\mathcal{R}z) \Rightarrow$ ((x divides y) and (y divides z)) $\Rightarrow$ (x divides z), then $x\mathcal{R}z$. Thus, $\mathcal{R}$ is transitive.*

3.4 Equivalence Relation

Definition 3.4.1. *Let $\mathcal{R}$ be a relation on a set A.*

1. $\mathcal{R}$ is called an equivalence relation if $\mathcal{R}$ is reflexive, symmetric, and transitive.

2. If $\mathcal{R}$ is an equivalence relation, then

(a) *For each $a \in A$, the set $\dot{a} = \{x \in A | x\mathcal{R}a\}$ is called the equivalence class of a modulo $\mathcal{R}$.*

(b) *The set $A_{/\mathcal{R}} = \{\dot{a} | a \in A\}$ is called the quotient set of A by $\mathcal{R}$.*

Examples 3.4.2. *1. The relation $\mathcal{R}$ given over $\mathbb{R}$ by the following formula $x\mathcal{R}y$ if and only if $x^2 = y^2$ is an equivalence relation, and $\dot{0} = \{0\}$, and for $a \neq 0$, $\dot{a} = \{a, -a\}$. $\mathbb{R}_{|\mathcal{R}} = \{\{0\},\ \{a, -a\},\ a > 0\}$.*

2. Let $\mathcal{R}_n$ be a relation of congruence modulo n defined on $\mathbb{Z}$ by $x\mathcal{R}_n y$ if and only if n divides $y - x$, is indeed an equivalence relation.
For this relation, we have

$$\begin{aligned}\dot{a} &= \{x \in \mathbb{Z}/n \text{ divides } x - a\}\\ &= \{x \in \mathbb{Z}/x = nq + a,\ q \in \mathbb{Z}\}\end{aligned}$$

noted $n\mathbb{Z} + a$.
In this case $\mathbb{Z}_{|\mathcal{R}_n} = \{n\mathbb{Z} + a, a \in \mathbb{Z}\}$ which is identified by $\mathbb{Z}_{|n\mathbb{Z}}$.

Remark 3.4.3. *The class $\dot{a}$ is also denoted as $\bar{a}$, $[a]$, and $Cl(a)$.*
If x is in an equivalence relation with y, we say that x and y are equivalent.

Theorem 3.4.4. *Let $\mathcal{R}$ be an equivalence relation on a non-empty set A, then*

1. *Every element of A is in an equivalence class. That is, $A = \cup\dot{a}$, where $a \in A$.*
2. *Two elements are equivalent if and only if they belong to the same class. That is, for all $a, x \in A$, $a\mathcal{R}x$ if and only if $\dot{a} = \dot{x}$.*
3. *Any equivalence classes are disjoint or coincide. That is, for all $a, x \in A$, $\dot{a} \cap \dot{x} = \emptyset$ or $\dot{a} = \dot{x}$.*
4. *The equivalence classes form a partition of A. That is, every element in A belongs to exactly one equivalence class, and the union of all equivalence classes covers A entirely.*

Proof :

1. Every element $a \in A$ verify $a\mathcal{R}a$, which means $a \in \dot{a}$.

2. Suppose that $a\mathcal{R}x$ and let $y \in \dot{a}$, then $y\mathcal{R}a$. Thus, by transitivity $z\mathcal{R}x$, so $y \in \dot{x}$. Then $\dot{a} \subset \dot{x}$. Similarly, $\dot{x} \subset \dot{a}$.

   Inversely, if $\dot{a} = \dot{x}$, we take an element $y \in \dot{a} = \dot{x}$, satisfies $a\mathcal{R}y$ and $y\mathcal{R}x$. Thus, by transitivity, we get $a\mathcal{R}x$.

3. Suppose the opposite, means $\dot{a} \cap \dot{x} \neq \emptyset$ and $\dot{a} \neq \dot{x}$. Thus, $\exists y \in A$ satisfies $a\mathcal{R}y$ and $y\mathcal{R}x$.Thus, by transitivity, we get $a\mathcal{R}x$ and using (2), we conclude that $\dot{a} = \dot{x}$, which is a contradiction with $\dot{a} \neq \dot{x}$.

4. Due to (1), we have $\dot{a} \neq \emptyset$ and $A = \cup\dot{a}$, where $a \in A$, and using (3) $\dot{a} \cap \dot{b} = \emptyset$ if $\dot{a} \neq \dot{b}$. Consequently, the equivalence classes form a partition of A.

□

Example 3.4.5. *If $n = 3$, we have* $\mathbb{Z}_{|3\mathbb{Z}} = \{\dot{0}, \dot{1}, \dot{3}\} = \{\dot{3}, \dot{1}, \dot{2}\} = \{\dot{-3}, \dot{4}, \dot{5}\}$.