3.6 Order Relation

Definition 3.6.1. *Let $\mathcal{R}$ be a relation on a set A.*

$\mathcal{R}$ is called an order relation if $\mathcal{R}$ is reflexive, antisymmetric, and transitive.

1. **(a)** *If $\mathcal{R}$ is an order relation, we often write $\leq_{\mathcal{R}}$ instead of $\mathcal{R}$.*

   (b) *$\leq_{\mathcal{R}}$ is called a total order relation if*

$$\forall x, y \in A,\ ((x \leq_{\mathcal{R}} y) \vee (y \leq_{\mathcal{R}} x)).$$

   (c) *$\leq_{\mathcal{R}}$ is called a partial order relation if*

$$\exists x, y \in A,\ ((x \not\leq_{\mathcal{R}} y) \wedge (y \not\leq_{\mathcal{R}} x)).$$

Remark 3.6.2. *Two elements x and y are said to be comparable by $\leq_{\mathcal{R}}$ if $x \leq_{\mathcal{R}} y$ or $y \leq_{\mathcal{R}} x$.*

Examples 3.6.3. 1. *The inclusion relation $\mathcal{R}$ on $\mathcal{P}(\mathbb{N})$ is a partial order. (For X, Y, and Z in $\mathcal{P}(\mathbb{N})$)*

   - *Reflexivity $X \subseteq X$*
   - *Antisymmetry $X \subset Y$ and $Y \subseteq X$ implies $X = Y$.*
   - *Transitivity $X \subseteq Y$ and $Y \subseteq Z$ implies $X \subseteq Z$. Moreover, the subsets $\{1, 2\}$ and $\{1, 3\}$ are incomparable.*

2. *The way words are arranged in a dictionary defines a total order relation called lexicographic order, denoted by $\leq_{lex}$. For example, algebra $\leq_{lex}$ analysis.*

Definition 3.6.4. *(Special Elements)Let $\mathcal{R}$ be an order relation on a set E, and let A be a subset of E, then*

1. *An element $m \in E$ is called a minimum of A if*

   (a) *$m \in A$.*

(b) *for every* $x \in A$*, we have* $m \leq_{\mathcal{R}} x$*. (We also say that* m *is a smallest (least) element of* A*.)*

2. *An element* $M \in E$ *is called a maximum of* A *if*

   (a) $M \in A$.

   (b) *for every* $x \in A$*, we have* $x \leq_{\mathcal{R}} M$*. (We also say that* M *is a greatest element of* A*.)*

3. *An extremum is an element that is either a minimum or a maximum.*

4. *An element* u *in the set* E *is called a lower bound of* A *if for every* $x \in A$*,* $u \leq_{\mathcal{R}} x$*. (It is also said that* A *is bounded below by* u*.)*

5. *An element* U *in the set* E *is called an upper bound of* A *if for every* $x \in A$*,* $x \leq_{\mathcal{R}} U$*. (It is also said that* A *is bounded above by* U*.)*

6. *The set* A *is said to be bounded below in* E *if* A *has a lower bound in* E*;* A *is said to be bounded above in* E *if* A *has an upper bound in* E*; and* A *is said to be bounded in* E *if* A *is both bounded below and above.*

7. *An element* w *in the set* E *is called a infimum of* A *if*

   (a) v *is a lower bound of* A*,*

   (b) *for every lower bound* v' *of* A*, we have* $v' \leq_{\mathcal{R}} v$*. Notation:* $v = inf(A)$.

8. *An element* V *in the set* E *is called a supremum of* A *if*

   (a) V *is an upper bound of* A*,*

   (b) *for every upper bound* $'V$ *of* A*, we have* $V \leq_{\mathcal{R}} V'$*. Notation:* $V = sup(A)$.

Examples 3.6.5. *For the usual order* $\leq$ *on the set of real numbers* $\mathbb{R}$*, let* $B =]-4, 0[\cup[\frac{1}{2}, +\infty[$*. We have the following*

There is no $min(B)$ *(*4 *is a good candidate, but* 4 *is not in* B*).*

There is no $max(B)$ *(There are no candidates for the maximum element in* B*).*

The set of lower bounds of B *is* $]-\infty, -4]$*, so* $inf(B) = -4$*.*

There are no upper bounds of B*, so* $sup(B)$ *does not exist.*