People's Democratic Republic of Algeria

University of science and technology Oran M-B — Academic year 2023/2024

Department of Mathematics — Algebra 1

Exercise series N°03

Exercise 1:

1. Are the following relations, reflexive? symmetric? antisymmetric? transitive?

   (a) $E = \mathbb{Z}$, and $x\mathcal{R}y \Leftrightarrow x = -y$;

   (b) $E = \mathbb{R}$, and $x\mathcal{R}y \Leftrightarrow cos^2(x) + sin^2(y) = 1$;

   (c) $E = \mathbb{N}$, and $x\mathcal{R}y \Leftrightarrow \exists p, q \geq 1,\ y = px^q (p, q \in \mathbb{N})$;

   (d) $E = \mathbb{R}$, and $x\mathcal{R}y \Leftrightarrow (x - y)(x^2 - y) = 0$.

2. Are these relations, order relations? or equivalence relations?

Exercise 2:

Let $\mathcal{R}$, the relation defined on $\mathbb{R}^*$ by

$$\forall x, y \in \mathbb{R}^*, x\mathcal{R}y \Leftrightarrow x - y = \frac{1}{x} - \frac{1}{y}.$$

1. Prove that $\mathcal{R}$ is an equivalence relation.

2. Determine the equivalence class of 2.

Exercise 3:

Let $\mathcal{R}$ a relation defined on $\mathbb{Z}$ by

$$x\mathcal{R}y \Leftrightarrow x^2 - y^2 = x - y.$$

1. Prove that $\mathcal{R}$ is an equivalence relation.

2. Determine the equivalence class of any element $x \in \mathbb{R}$. Precise the equivalence class of 1.

Exercise 4:

Let $\mathcal{R}$ a relation defined on $\mathbb{Z}$ by

$$x\mathcal{R}y \Leftrightarrow x - y \text{ is multiple of 3.}$$

1. Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

2. Determine the quotient set.

3. Prove that $\dot{1\bar{5}} = \dot{0}$ and $\dot{9} \cap \dot{5\bar{8}} = \emptyset$.

Exercise 5:

Let $\mathcal{R}$ a relation defined on $\mathbb{Z}$ by

$$x\mathcal{R}y \Leftrightarrow \exists k \in \mathbb{Z},\ x - y = 4k.$$

1. Prove that $\mathcal{R}$ is an equivalence relation.

2. Determine $\mathbb{Z}_{|\mathcal{R}}$.

Exercise 6:

Let E be a non empty set, and $\Gamma \subset \mathcal{P}(E)$ non empty, satisfies the following property:

$$\forall X, Y \in \Gamma,\ \exists Z \in \Gamma,\ Z \subset (X \cap Y).$$

We define on $\mathcal{P}(E)$ the following relation $A\mathcal{R}B \ \Leftrightarrow\ \exists X \in \Gamma,\ X \cap A = X \cap B$.

Prove that $\mathcal{R}$ is an equivalence relation on $\mathcal{P}(E)$. What are the equivalence classes of $\emptyset$ and E?

Exercise 7:

Let $\mathcal{R}$ a relation defined on $\mathbb{R}$ by

$$x\mathcal{R}y \ \Leftrightarrow\ x^3 - y^3 \geq 0.$$

Prove that $\mathcal{R}$ is an order relation. Is the order total?

Exercise 8:

Let $\prec$ a relation defined on $\mathbb{R}^2$ by

$$(x, y) \prec (x', y') \ \Leftrightarrow\ x \leq x' \text{ and } y \leq y'.$$

1. Prove that $\prec$ is an order relation. Is the order total?

2. The closed disk of center 0 and radius 1, have upper bounds? greatest element 'max'? supremum?