Exercise series N°03

Exercise 1:

1. Are the following relations, reflexive? symmetric? antisymmetric? transitive?

   (a) $E = \mathbb{Z}$, and $x\mathcal{R}y \;\Leftrightarrow\; |x| = |y|$;

   (b) $E = \mathbb{R}$, and $x\mathcal{R}y \;\Leftrightarrow\; cos^2(x) + sin^2(y) = 1$;

   (c) $E = \mathbb{R}$, and $x\mathcal{R}y \;\Leftrightarrow\; (x-y)(x^2-y) = 0$.

2. Are these relations, order relations? or equivalence relations?

Exercise 2:

Let $\mathcal{R}$, the relation defined on $\mathbb{Z}$ by

$$\forall x, y \in \mathbb{Z}^2, x\mathcal{R}y \;\Leftrightarrow\; x - y \; is \; a \; multiple \; of \; 3.$$

1. Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

2. Determine the quotient set.

3. Prove that $\overline{9} = \overline{0}$, $\overline{12} \cap \overline{10} = \emptyset$.

Exercise 3:

Let $\mathcal{R}$ a relation defined on $\mathbb{R}$ by

$$x\mathcal{R}y \;\Leftrightarrow\; x^2 - y^2 = x - y.$$

1. Prove that $\mathcal{R}$ is an equivalence relation.

2. Determine the equivalence class of any element $x \in \mathbb{R}$. Precise the equivalence class of 1.

Exercise 4:

We define on $\mathbb{R}^2$ the relation $\mathcal{R}$ by:

$$(x, y)\mathcal{R}(x', y') \;\Leftrightarrow\; x + y = x' + y'.$$

1. Prove that $\mathcal{R}$ is an equivalence relation.

2. Determine the equivalence class of the couple $(0, 0)$.

Exercise 5:

Let $\prec$ a relation defined on $\mathbb{R}^2$ by

$$(x, y) \prec (x', y') \;\Leftrightarrow\; x \leq x' \; and \; y \leq y'.$$

1. Prove that $\prec$ is an order relation. Is the order total?

2. The closed disk of center 0 and radius 1, have upper bounds? greatest element 'max'? supremum?

Exercise 6:

Let $\prec$ be a relation defined on $E = \{(x, y) \in \mathbb{R}^2,\ x \leq y\}$ by

$$(x, y) \prec (x', y') \ \Leftrightarrow \ (x, y) = (x', y') \ \text{or} \ y \leq x'.$$

Prove that $\mathcal{R}$ is an order relation on E.

Exercise 7:

Let $\prec$ a relation defined on $]1, +\infty[$ by

$$x \prec y \ \Leftrightarrow \ \frac{x}{1+x^2} \geq \frac{y}{1+y^2}.$$

Prove that $\prec$ is a total order relation.