Exercise series N°04

Exercise 1:

Show that the given laws equip the set G with a group structure and determine if it's abelian

1. $x * y = x^{ln(y)}$ on $G = \{x \in \mathbb{R},\ x > 0\ and\ x \neq 1\}$;
2. $(x, y) * (x', y') = (x + x', ye^{x'} + y'e^{-x})$ on $G = \mathbb{R}^2$.

Exercise 2:

In the following cases, determine if H is a subgroup of group G

1. $G = (\mathbb{Z}, +)$; $H =$\{even numbers\}.
2. $G = (\mathbb{R}, +)$; $H = [-1, +\infty[$.
3. $G = (\mathbb{R}^*, \times)$; $H = \mathbb{Q}^*$.

Exercise 3:

Let the following elements in S_4

$$id = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},\quad M_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},\quad M_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},\quad M_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

Compute $M_1 \circ M_3$, $M_1 \circ M_1$, $M_1 \circ M_2$ and $M_2 \circ M_1$.

Exercise 4:

Let $(G, *)$ be a group, we consider H subset of G defined by

$$H = \{h \in G, \forall g \in G,\ g * h = h * g\}.$$

1. Show that $(H, *)$ is a subgroup of G.
2. If G is a commutative group, what is H?

Exercise 5:

1. Let $G = \mathbb{R}$, and $*$ is a binary operation on G defined by

$$\forall x, y \in G,\ x * y = x + y - \frac{1}{2}.$$

   (a) Prove that $*$ is a commutative operation.

   (b) Prove that $(G, *)$ is group.

2. Let

$$f : (G, *) \to (\mathbb{R}, +)$$
$$x \mapsto f(x) = 2x - 1.$$

   (a) Prove that f is a group homomorphism.

   (b) Is f bijective? Justify.

Exercise 6:

Let $\mathbb{D}$ be the set of decimal numbers: $\mathbb{D} = \left\{ \frac{n}{10^k} \;\middle|\; n \in \mathbb{Z},\ k \in \mathbb{N} \right\}$. Show that $\mathbb{D}$ is a subring of $(\mathbb{R}, +, \times)$.

Exercise 7:

We consider $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2};\ a, b \in \mathbb{Z}\}$.

1. Show that $(\mathbb{Z}[\sqrt{2}], +, \times)$ is a ring.
2. We denote $N(a + b\sqrt{2}) = a^2 - 2b^2$. Show that, for all x, y from $\mathbb{Z}[\sqrt{2}]$, we have $N(xy) = N(x)N(y)$.
3. Deduce that the invertible elements of $\mathbb{Z}[\sqrt{2}]$ are those that can be written as $a + b\sqrt{2}$ with $a^2 - 2b^2 = \pm 1$.

Exercise 8:

Let p be a prime number. We define

$$\mathbb{Z}_p = \{\frac{a}{b},\ (a, b) \in \mathbb{Z} \times \mathbb{Z}, p \wedge b = 1\}.$$

1. Prove that $\mathbb{Z}_p$ is a sub-ring of $(\mathbb{Q}, +, \times)$.
2. Prove that for any nonzero rational number x, $x \in \mathbb{Z}_p$, or $x^{-1} \in \mathbb{Z}_p$.
3. Let B be a sub-ring of $\mathbb{Q}$ containing $\mathbb{Z}_p$. Show that $B = \mathbb{Q}$ or $B = \mathbb{Z}_p$.