Chapter 1

Vector spaces

1.1 External operation

Given a field $\mathbb{F}$. We call an external operation on a non empty set E with coefficients in $\mathbb{F}$, every application $\bullet$ from $\mathbb{F} \times E$ to E

$$\begin{aligned} \bullet : \mathbb{F} \times E &\longrightarrow E \\ (\alpha, x) &\longmapsto \alpha \bullet x \end{aligned}$$

Example 1.1.1. *The application $\bullet$ from $\mathbb{R} \times \mathbb{C}$ to $\mathbb{C}$, defined by $\alpha \bullet z = \alpha z$, is an external composition law on $\mathbb{C}$ with coefficients in the field $\mathbb{R}$.*

Example 1.1.2. *The application $\bullet$ from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$, defined by $\alpha \bullet (x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$, is an external composition law on $\mathbb{R}^n$ with coefficients in the field $\mathbb{R}$.*

1.2 Structure of Vector Space

Definition 1.2.1. *Let $\mathbb{F}$ be a commutative field. A vector space over $\mathbb{F}$ (or $\mathbb{F}$-vector space) is any non-empty set E equipped with two operations, an internal one denoted by $+$ and an external one denoted by $\bullet$, with coefficients in $\mathbb{F}$, satisfying:*

1. *$(E, +)$ is an abelian group.*

2. *For all $\alpha, \beta \in \mathbb{F}$, $u, v \in E$:*

   - $\alpha \bullet (u + v) = \alpha \bullet u + \alpha \bullet v$.
   - $(\alpha + \beta) \bullet u = \alpha \bullet u + \beta \bullet u$.
   - $(\alpha\beta) \bullet u = \alpha \bullet (\beta \bullet u)$.
   - $1 \bullet u = u$, *where* 1 *is the identity element of* $\mathbb{F}$.

The elements of E are called vectors, and the elements of K are called scalars. In all the examples that follow, the verification of axioms is straightforward and is left to the students. Only the neutral element values of the internal operation and the symmetric of an element will be indicated in each case.

Example 1.2.2. *Let's set* $\mathbb{F} = \mathbb{R}$ *and* $E = \mathbb{R}^2$. *By the definition of the Cartesian product, we have if* (x, y) *and* (x', y') *are elements of* $\mathbb{R}^2$ *and* $\alpha \in \mathbb{R}$, *then addition and scalar multiplication are defined as follows:*

$$(x, y) + (x', y') = (x + x', y + y'),$$

$$\alpha \bullet (x, y) = (\alpha x, \alpha y).$$

The neutral element of the internal law is the zero vector $(0, 0)$. *The inverse of* (x, y) *is* $(-x, -y)$, *also denoted as* $-(x, y)$

Example 1.2.3. *Let* n *be an integer greater than or equal to* 1. *Set* $\mathbb{F} = \mathbb{R}$ *and* $E = \mathbb{R}^n$. *If* $(x_1, x_2, ..., x_n)$ *and* $(y_1, y_2, ..., y_n)$ *are elements of* $\mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$, *then addition and scalar multiplication are defined as follows:*

$$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n),$$

$$\alpha \bullet (x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n).$$

The neutral element of the internal law is the zero vector $(0, 0, ..., 0)$*. The inverse of* $(x_1, x_2, ..., x_n)$ *is* $(-x_1, -x_2, ..., -x_n)$*.*

Example 1.2.4. *The set of functions* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is denoted as* $\mathcal{F}(\mathbb{R}, \mathbb{R})$*. It is endowed with the structure of an* $\mathbb{R}$*-vector space as follows: Let* f *and* g *be elements of* $\mathcal{F}(\mathbb{R}, \mathbb{R})$*, and let* α *be a real number. Let* $x \in \mathbb{R}$*, the internal operation* $+$ *and the external operation* $\bullet$ *are defined by the given expressions,*

$$(f + g)(x) = f(x) + g(x),$$

$$(\alpha \bullet f)(x) = \alpha \times f(x).$$

The neutral element for addition is the zero function $f(x) = 0$*, for all* x *in* $\mathbb{R}$*. The additive inverse of the element* f *in* $\mathcal{F}(\mathbb{R}, \mathbb{R})$ *is the function* $-f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$*.*

Theorem 1.2.5. *Let* E *be an* $\mathbb{F}$*-vecor space, then for all* $u, v \in E$ *and* $\alpha, \beta \in \mathbb{F}$*, we have*

1. $0_{\mathbb{F}} \bullet x = 0_E = \alpha \bullet 0_E$,
2. $\alpha \bullet (-u) = -\alpha \bullet u = (-\alpha) \bullet u$,
3. $\alpha \bullet (u - v) = \alpha \bullet u - \alpha \bullet v$ *and* $(\alpha - \beta) \bullet u = \alpha \bullet u - \beta \bullet u$,
4. *If* $\alpha \bullet u = 0_E$ *then* $\alpha = 0_{\mathbb{F}}$ *or* $u = 0_E$*.*

1.3 Linear combination

Definition 1.3.1. *Let* n *be an integer greater or equal to* 1*, and let* $v_1, v_2, ..., v_n$ *be vectors in a vector space* E*. Any vector of the form* $u = a_1v_1 + a_2v_2 + ... + a_nv_n$*, where* $a_i \in \mathbb{F}$*, is called a linear combination of the vectors* $v_1, v_2, ..., v_n$*. The scalars* $a_1, a_2, ..., a_n$ *are called coefficients of the linear combination.*

Remark 1.3.2. *In the case of $n = 1$, then $u = av$, and u is said to be colinear with v.*

Example 1.3.3. *In the $\mathbb{R}$-vector space $\mathbb{R}^3$, $(3, 3, 1)$ is a linear combination of the vectors $(1, 1, 0)$ and $(1, 1, 1)$ because we have the equality*

$$(3, 3, 1) = 2(1, 1, 0) + (1, 1, 1).$$

1.4 Vector Subspace

A subset F of a $\mathbb{F}$-vector space E is called a vector subspace if it is non-empty and itself forms a $\mathbb{F}$-vector space with the operations restricted to F: internal addition $(+)$ and external scalar multiplication $(\bullet)$.

Proposition 1.4.1. *A subset F of vector space E is a vector subspace if and only if:*

1. *F contains the zero element 0_E.*
2. *For all $a \in \mathbb{F}$ and all $x, y \in F$ $x - y \in F$ and $\alpha \bullet x \in F$.*

Proof:

a) Suppose F is a vector subspace of E under the restricted operations. The external operation of E induces an external operation on F, so for all $\alpha \in \mathbb{F}$ and all $x \in F$, we have $\alpha \cdot x \in F$. Regarding internal addition $(+)$, $(F, +)$ is a subgroup of E, hence $0_E \in F$, and for all x and y in F, we have $x - y \in F$.

b) Suppose F satisfies assertions 1) and 2). Then, $(F, +)$ is a subgroup of the commutative group $(E, +)$. Assertion 2) shows that the external scalar multiplication of E induces an external scalar multiplication on F, and the fact that $(F, \cdot)$ is a subgroup ensures assertion 2) of definition 1.2.1. Consequently, F is a vector space under the restricted operations of E, i.e., a vector subspace of E.

Corollary 1.4.2. *A subset F of E is a vector subspace of the $\mathbb{F}$-vector space E if and only if:*

1. *F contains the zero element 0_E.*
2. *For all $\alpha, \beta \in \mathbb{F}$ and all $x, y \in F$, $\alpha x + \beta y \in F$.*

Examples 1.4.3.

1. *$F = \{(x, y, z) \in \mathbb{R}^3 : 2x + y - 3z = 0\}$ is a subspace of $\mathbb{R}^3$.*
2. *$F = \{(x, y, z) \in \mathbb{R}^3 : y - 2z - 2 = 0\}$ is not a subspace of $\mathbb{R}^3$. Indeed, $(0, 0, 0)$ is not in F.*
3. *$F = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$ is a subspace of $\mathbb{R}^2$.*

1.5 Direct Sums

Theorem 1.5.1. *The finite sum of subspaces of a $\mathbb{F}$-vector space E is a subspace of E. In other words, if $F_1, F_2, ..., F_n$ is a finite family of subspaces of a $\mathbb{F}$-vector space E, then $\sum_{i=1}^{n} F_i$ is a subspace of E.*

Proof:

1. For every $i \in \{1, 2, ..., n\}$, we have $0_E \in F_i$, so $0_E = 0_E + 0_E + ... + 0_E \in \sum_{i=1}^{n} F_i$ (n times).
2. If $a, b \in \sum_{i=1}^{n} F_i$ and $\alpha, \beta \in \mathbb{F}$, then $a = \sum_{i=1}^{n} a_i$ and $b = \sum_{i=1}^{n} b_i$, with $a_i, b_i \in F_i$. We have $\alpha \bullet a + \beta \bullet b = \alpha \bullet \sum_{i=1}^{n} a_i + \beta \bullet \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (\alpha \bullet a_i + \beta \bullet b_i)$. Then, $\alpha \bullet a + \beta \bullet b \in \sum_{i=1}^{n} F_i$. Consequently, $\sum_{i=1}^{n} F_i$ is a subspace of E.

Definition 1.5.2. *A sum $F = F_1 + \ldots + F_n$ is said to be direct sum, if every vector u in F can be uniquely expressed as $u = u_1 + \ldots + u_n$, where $u_i \in F_i$ for $i = 1, 2, \ldots, n$. It is denoted as $F = F_1 \oplus \ldots \oplus F_n$.*

Example 1.5.3. *Let F_1 and F_2 be subspaces of $\mathbb{R}^3$ defined by*

$$F_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

$$F_1 = \{(x, y, z)\mathbb{R}^3 \mid x - z = 0\}.$$

The sum $F_1 + F_2$ is not direct since we have the following inequality

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

We have $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \in F_1$ and $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \in F_2$, we also have $\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \in F_1$ and $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} F_2$.
The decomposition of an element of $F_1 + F_2$ as summation of an element in F_1 and an element in F_2 is not unique. Then, the sum is not direct.

Definition 1.5.4. *The vector space E is said to be the direct sum of subspaces E_1 and E_2, or that the subspaces are complementary in E, when $E_1 + E_2 = E$ and $E_1 \cap E_2 = \{0_E\}$. In this case, it is denoted as $E = E_1 \oplus E_2$.*

Example 1.5.5. *Let $E = \mathbb{R}^2$ and F_1, F_2 be subspaces of $\mathbb{R}^2$ defined by*

$$F_1 = \mathbb{R} \times \{0\} = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$$

and

$$F_2 = \{0\} \times \mathbb{R} = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}.$$

It is evident that $E = F_1 + F_2$, and the intersection is $\{0_{\mathbb{R}^2}\}$, so $E = F_1 \oplus F_2$.

1.6 Intersection of Vector Subspaces

Proposition 1.6.1. *Let F, G be vector subspaces of a vector space E over field $\mathbb{F}$. The intersection $F \cap G$ is a vector subspace of E. More generally, if $(F_i)_{i \in I}$ is a family of vector subspaces of E, then the set $\cap F_i$ is a subspace of E.*

Example 1.6.2. *The following sets F_1, F_2 defined below are vector subspaces of $\mathbb{R}^3$*

$$F_1 = \{0\} \times \mathbb{R} \times \mathbb{R} = \{(0, y, z) \in \mathbb{R}^3, y, z \in \mathbb{R}\},$$

$$F_2 = \mathbb{R} \times \{0\} \times \mathbb{R} = \{(x, 0, z) \in \mathbb{R}^3, x, z, \in \mathbb{R}\}.$$

$F_1 \cap F_2$ is a vector subspace of $\mathbb{R}^3$.

Remark 1.6.3. *It should be noted that the union of vector subspaces of E is not necessarily a vector subspace of E. For example, consider the two subspaces E_1 and E_2 of the product space $\mathbb{R}^2$ defined by*

$$E_1 = \mathbb{R} \times \{0\}, \quad E_2 = \{0\} \times \mathbb{R}.$$

The set $E_1 \cup E_2$ is not a subspace of $\mathbb{R}^2$ since $(1, 0) \in E_1$ and $(0, 1) \in E_2$. However, $(1, 0) + (0, 1) = (1, 1) \notin E_1 \cup E_2$.