1.7 Spanning vector spaces

Definition 1.7.1. *Let $\mathcal{F} = \{v_1, v_2, ..., v_n\}$ be a family of vectors in an $\mathbb{F}$-vector space E. The vector subspace spanned by $\mathcal{F}$, denoted as vect$(\mathcal{F})$ (or span$(\mathcal{F})$), is defined as the set of all linear combinations of elements in $\mathcal{F}$, and we have*

$$u \in vect(\mathcal{F}) \;\Leftrightarrow\; \exists \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{F} : \; u = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n.$$

Proposition 1.7.2. *Let $\mathcal{G}$ be a family of vectors in E, and F be a vector subspace of E. The following properties are equivalent:*

1. *F is the smallest vector subspace containing $\mathcal{G}$.*

2. *F is the intersection of all vector subspaces containing $\mathcal{G}$.*

3. *F is the set of all linear combinations of vectors in $\mathcal{G}$, i.e., $F = vect(\mathcal{G})$.*

Example 1.7.3.

1. *In $\mathbb{R}^3$, let $u_1 = (1, 0, 0)$ and $u_2 = (0, 1, 0)$, then $vect(u_1, u_2) = \{(x, y, 0),\; x, y \in \mathbb{R}\}$.*

2. *Let $E_1 = \{(x, y, z) \in \mathbb{R}^3,\; x + y - z = 0\}$, we have*

$$E_1 = \{x(1, 0, 1) + y(0, 1, 1),\; x, y \in \mathbb{R}\} = vect((1, 0, 1), (0, 1, 1)).$$

Which means that E_1 is a subspace of $\mathbb{R}^3$ generated by $(1, 0, 1), (0, 1, 1)$.

1.8 Linearly independent, dependents and generating sets, bases

Definition 1.8.1. *Let E be an $\mathbb{F}$-vector space and $\mathcal{B} = (u_1, u_2, ..., v_n)$ a family of vectors of E.*

1. *The family $\mathcal{B}$ is said to be linearly dependant if there exists scalars $\lambda_{1,}, \lambda_2, ..., \lambda_n$ not all zeros, in $\mathbb{F}$ satisfying $\lambda_1 u_1 + \lambda_2 u_2 + ... + \lambda_n u_n = 0$.*

2. *The family $\mathcal{B}$ is said to be linearly independent if*

$$\lambda_1 u_1 + \lambda_2 u_2 + ... + \lambda_n u_n = 0 \ \Rightarrow \ \lambda_i = 0, i = 1, ..., n.$$

Example 1.8.2. 1. *Let the family $\mathcal{B} = \{1, i\}$ in $\mathbb{R}$-vector space $\mathbb{C}$, $\mathcal{B}$ is a linearly independent family, since $\forall \alpha, \beta \in \mathbb{R}$, $\alpha(1) + i\beta = 0$ implies $\alpha = \beta = 0$.*

2. *Let the family $\mathcal{B} = \{1, X, X^2\}$ in $\mathbb{R}$-vector space $\mathbb{R}_2[X]$ (the set of polynomials of degree less or equal to 2).*
   Let $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha(1) + \beta(X) + \gamma(X^2) = 0$ implies $\alpha = \beta = \gamma = 0$. Which means $\mathcal{B}$ is a linearly independent set in $\mathbb{R}_2[X]$.

3. *The set $\mathcal{B} = \{(1, 2, -1), (3, 0, 1), (0, -6, 4)\}$ is linearly dependent in $\mathbb{R}^3$. Since $\alpha(1, 2, -1) + \beta(3, 0, 1) + \gamma(0, -6, 4) = 0_{\mathbb{R}^3}$, implies $\alpha = 3$, $\beta = -1$ and $\gamma = 1$.*

Definition 1.8.3. *Let E be an $\mathbb{F}$-vector space and $\mathcal{B}$ a family of vectors in E, $\mathcal{B}$ is said generating family or span of E if $E = span(\mathcal{B})$. In other words, any vector of E can be written as a linear combination of vectors in E.*

Examples 1.8.4. 1. *Let in $\mathbb{R}^3$, the family of vectors $\mathcal{B} = \{e_1(1, 0, 0), e_2(0, 1, 0), e_3(0, 0, 1)\}$. Then, $\mathbb{R}^3 = span\{\mathcal{B}\}$, since $\forall u(x, y, z) \in \mathbb{R}^3$: $u = xe_1 + ye_2 + ze_3$.*

2. *Let in $\mathbb{R}^n$, the family of vectors $\mathcal{B} = \{e_1(1, 0, ..., 0), e_2(0, 1, ..., 0), ..., e_n(0, ..., 0, 1)\}$. Hence, $\mathbb{R}^n = span\{\mathcal{B}\}$.*

Definition 1.8.5. *We say that a family $\mathcal{B} = \{u_1, ..., u_n\}$ of a vector space, basis of E if any vector of E can be written uniquely as linear combination of vectors in $\mathcal{B}$. In other words,*

$$\forall u \in E, \ \exists! \alpha_1, ..., \alpha_n \in \mathbb{F}, \ \text{such that } u = \alpha_1 u_1 + ... + \alpha_n u_n.$$

Proposition 1.8.6. *Let E be a vector space, and $\mathcal{B}$ a family of vectors in E. We say that $\mathcal{B}$ is a basis for E if and only if, it is linearly independent and generating family of E. On note $dim E$, the dimension of the vector space E and we have $dim(E) = card(\mathcal{B})$, where $\mathcal{B}$ is a basis for E.*

Examples 1.8.7.

1. *Let in $\mathbb{R}^3$, the subspace $S = span\{(1,-4,-3)^t, (-3,6,7)^t, (-4,-2,6)^t\}$, the basis of S is $\{(1,-4,-3)^t, (-3,6,7)^t\}$, since it is a linearly independent and a generating set for S.*

2. *Let in $\mathbb{R}^3$, the subspace $S = \{(x,y,z) \in \mathbb{R}^3,\ x + 2y - z = 0\}$. We have for any $u(x,y,z) \in S$, $z = x + 2y$, which means*

$$(x,y,z) = (x,y,x+2y) = x(1,0,1) + y(0,1,2),\ x,y \in \mathbb{R}.$$

   Then, $S = span\{(1,0,1),(0,1,2)\}$ and since $(1,0,1),(0,1,2)$ are linearly independent, then $\{(1,0,1),(0,1,2)\}$ is a basis for S, $dim(S) = 2$.

Theorem 1.8.8. *Let $\mathcal{B} = \{u_1, ..., u_n\}$ be a finite subset of a $\mathbb{F}$-vector space E. Then the following statements are equivalent:*

1. *$\mathcal{B}$ is a basis of E.*

2. *$\mathcal{B}$ is a maximal linearly independent set in E.*

3. *$\mathcal{B}$ is a minimal spanning set for E.*

4. *Every $u \in E$ can be uniquely written as $u = \alpha_1 u_1 + ... + \alpha_n u_n$, with $\alpha_1, ..., \alpha_n \in \mathbb{F}$.*

Corollary 1.8.9. *Let E_1 and E_2 be two finite sets of an $\mathbb{F}$-vector space E such that E_1 is linearly independent, and E_2 is a generating set, and $E_1 \subset E_2$. Then, there exists a basis A of E such that $E_1 \subset A \subset E_2$.*

Corollary 1.8.10. *(Theorem of incomplete basis). If E is an $\mathbb{F}$-vector space of finite dimension and E_1 is a linearly independent family of vectors of E, then there exists a basis A of E, such that $E_1 \subset A$.*

Examples 1.8.11. 1. *$dim_{\mathbb{R}}\mathbb{C} = 2$, since $\{1, i\}$ is a basis of $\mathbb{C}$ as a vector space over $\mathbb{R}$.*

$$\forall z \in \mathbb{C},\ z = x(1) + y(i),\ x, y \in \mathbb{R}.$$

2. *$dim_{\mathbb{C}}\mathbb{C} = 1$, since $\{1\}$ is a basis of $\mathbb{C}$ as a vector space over $\mathbb{C}$.*

$$\forall z \in \mathbb{C},\ z = z(1).$$

3. *$dim_{\mathbb{B}}\mathbb{R}^n = n$, since $\{e_1, e_2, ..., e_n\}$ is a basis of $\mathbb{R}^n$ as a vector space over $\mathbb{R}$.*

Example 1.8.12. *Let $E = span\{u = (1, -4, -3)^t, v = (-3, 6, 7)^t, w = (-4, -2, 6)^t\}$, find a basis for E.*
Let $\mathcal{B} = \{(1, -4, -3)^t, (-3, 6, 7)^t, (-4, -2, 6)^t\}$, $\mathcal{B}$ is generating. However, the vectors are linearly dependents, since, there exists $\alpha, \beta, \gamma \in \mathbb{R}$, such that $\alpha u + \beta v + \gamma w = 0_{\mathbb{R}^3}$. Choose $\alpha = -5$, $\beta = -3$ and $\gamma = 1$.
Consider $\mathcal{B}' = \{u = (1, -4, -3)^t, v = (-3, 6, 7)^t\}$, $\mathcal{B}'$ is linearly independent and generating family for E, then $\mathcal{B}'$ is a basis of E and $dimE = 2$.

Exercise 1.8.13. *Consider the subspace E of $\mathbb{R}^3$ defined by:*

$$E = \{(x, y, z) \in \mathbb{R}^3,\ x + y - 2z = 0 \wedge x - 3z = 0\}.$$

1. *Determine a basis of E and precise its dimension.*
2. *Determine $dimF$, where F is a subspace of $\mathbb{R}^3$ spanned by the vectors*

$$u_1(1, 2, 2), u_2(0, 6, -2), u_3(1, 5, 1).$$

3. *Determine* $E \cap F$.

4. *Is* $\mathbb{R}^3 = E \oplus F$*? justify.*