Exercise series N°01

Exercise 01:

Let $\mathbb{R}[X]_{\leq n}$ the set of polynomials of degree less or equal to n with real coefficients, and the operations, addition " $+$ " of polynomials, and scalar multiplication of polynomials by real number, noted " $\bullet$ ".

1. Show that $(\mathbb{R}[X]_{\leq n}, +, \bullet)$ is an $\mathbb{R}$-vector apace.
2. What happen if we consider $\mathbb{R}[X]_{=n}$?
3. The set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, noted $\mathcal{F}(\mathbb{R}, \mathbb{R})$ with the operation:
   - Internal operation " $+$ ": let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, then $\forall x \in \mathbb{R}, \quad (f+g)(x) = f(x) + g(x)$.
   - External operation " $\bullet$ ": Let $\alpha \in \mathbb{R}$, and let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, then $\forall x \in \mathbb{R}, \quad (\alpha \bullet f)(x) = \alpha \bullet f(x)$.
   Show that $(\mathcal{F}(\mathbb{R}, \mathbb{R}), +, \bullet)$ is an $\mathbb{R}$-vector space.

Exercise 02:

Let $E = \mathbb{R}^2$ with the operations, an internal operation $+$ and an external one noted $\bullet$ defined by

$$\forall (x,y), (x',y') \in \mathbb{R}^2, \forall \alpha \in \mathbb{R} : \; (x,y) + (x',y') = (x+x', y+y'), \; \alpha \bullet (x,y) = (\alpha x, 0).$$

Is $(E, +, \bullet)$ an $\mathbb{R}$-vector space? Justify.

Exercise 03:

Say whether the following sets are vector-subspaces of E or not.

1. $E_1 = \{(x,y,z) \in \mathbb{R}^3, \; x + 2y - z = 0\}, \; E = \mathbb{R}^3$.
2. $E_2 = \{(x,y) \in \mathbb{R}^2, \; xy = 0\}, \; E = \mathbb{R}^2$.
3. $E_3 = \{P \in \mathbb{R}_2[X], \; P + P' = 0\}, \; E = \mathbb{R}_2[X]$.
4. $E_4 = \{(x,y,z) \in \mathbb{R}^3, \; x + 2y - z = 0\} \cap \{(x,y,z) \in \mathbb{R}^3, \; x - y + z = 0\}, \; E = \mathbb{R}^3$.

Exercise 04:

- In the following cases, is the vector U a linear combination of the vectors U_i:
  1. $E = \mathbb{R}^2, \; U = (1,2)^t, \; U_1 = (1,-2)^t, \; U_2 = (2,3)^t$.
  2. $E = \mathbb{R}^2, \; U = (1,2)^t, \; U_1 = (1,-2)^t, \; U_2 = (2,3)^t, \; U_3 = (-4,5)^t$.
  3. $E = \mathbb{R}^3, \; U = (2,5,3)^t, \; U_1 = (1,3,2)^t, \; U_2 = (1,-1,4)^t$.
- In the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$, are the function $x \longmapsto e^x$ and $x \longmapsto sin(x)$ linear combinations of the functions $x \longmapsto cos(x)$ and $x \longmapsto sin(x)$

Exercise 05:

Let $E = \mathcal{F}(\mathbb{R}, \mathbb{R})$ the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, we consider F the vector subspace of the even functions, and G the subspaces of the odd functions. Prove that $E = F \oplus G$.

Exercise 06:

1. Are the following vectors form a linearly independent, generating family in $\mathbb{R}^3$?

   $$B_1 = \{(-1,1,2)^t, (4,2,5)^t\}, \; B_2 = \{(1,2,0)^t, (2,0,0)^t, (0,1,0)^t, (1,1,1)^t\}, B_3 = \{(1,1,0)^t, (4,1,4)^t, (2,-1,4)^t\}.$$

2. Let in $\mathbb{R}^4$, the following vectors $U = (1,2,3,4)^t$and $V = (1,-2,3,-4)^t$. Can you find α and β such that $(\alpha, 1, \beta, 1)^t \in Vect\{U,V\}$, and $(\alpha, 1, 1, \beta)^t \in Vect\{U,V\}$.

Exercise 07:

Prove that the vectors $U = (0,1,1)^t$, $V = (1,0,1)^t$ and $W = (1,1,0)^t$ form a basis for $\mathbb{R}^3$. Find the coordinates of the vector $(1,1,1)^t$ in this basis.

Exercise 08:

1. Consider F be a subset of $\mathbb{R}^3$ defined by

$$F = \{(x,y,z) \in \mathbb{R}^3,\ z = -2x\}$$

   Prove that F is a vector-subspace of $\mathbb{R}^3$ and determine $dimF$.

2. Let $U = (-1,0,2)^t$, $V = (1,2,0)^t$ and $W = (2,2,-2)^t$
   Determine the vector-subspace G generated by $\{U,V,W\}$, as well as its dimension.

3. Determine $F \cap G$ and its dimension.

4. Is $\mathbb{R}^3 = F \oplus G$? Justify.

Exercise 09:

Determine the rank of the family F in $\mathbb{R}^3$, where $F = \{U_1, U_2, U_3, U_4\}$, and $U_1 = (1,0,2)^t$, $U_2 = (2,1,1)^t$, $U_3 = (-1,-2,4)^t$ and $U_4 = (2,-1,7)^t$.

Supplementary Exercises

Exercise 10:

Let $E = \{(x,y,z,t) \in \mathbb{R}^4,\ x - t = 0\ and\ y + z = 0\}$ and $F =< \{U,V,W\} >$, where $U = (1,-1,1,1)^t$, $V = (1,1,-1,1)^t$, $W = (1,1,1,-1)^t$.

1. Prove that E is a vector subspace of $\mathbb{R}^4$.

2. Determine the subspace $E + F$, is this summation direct.

Exercise 11: Let $E = \{P \in \mathbb{R}_2[X],\ P(1) = 0\}$

1. Prove that E is a vector-subspace of $\mathbb{R}_2[X]$.

2. Determine a basis for E, and conclude its dimension.

Exercise 12:

In the vector space $\mathbb{R}^3$, we consider the following vector-subspaces:

$$F = \{(x,y,z) \in \mathbb{R}^3,\ x + y + z = 0\},\ G = \{(x,y,z) \in \mathbb{R}^3,\ x = y = z\}\ and\ H = \{(x,y,z) \in \mathbb{R}^3,\ x = y = 0\}.$$

1. Find a basis for each of the previous vector-subspace.

2. Is $\mathbb{R}^3 = F \oplus G$? Is $\mathbb{R}^3 = F \oplus H$? What do you conclude?