Linear applications

Notations

$\mathcal{L}(E, F)$: denotes the set of linear applications from E to F.

We denote the subspace $Null(f)$ as $Ker f$ which is $\{x \in E,\ f(x) = 0_F\}$.

We denote $Range(f)$ as $Im f$ which is $\{f(x),\ x \in E\}$.

0.1 Homomorphisms of Vector Spaces

Definition 0.1.1. *Let E and F be $\mathbb{F}$-vector spaces. A linear application (or homomorphism of vector spaces) from E to F is any function $f : E \longrightarrow F$ such that for all x, y in E and for any λ in $\mathbb{F}$, the following hold: $f(x + y) = f(x) + f(y)$ and $f(\lambda \bullet x) = \lambda \bullet f(x)$.*
A bijective linear application is called an isomorphism.
A linear application from E to itself is called an endomorphism of E. A bijective endomorphism is called an automorphism.

Proposition 0.1.2. *Let E and F be two $\mathbb{F}$-vector spaces. An application $f : E \longrightarrow F$ is a linear application if and only if, for all x, y in E and for all $\alpha, \beta \in \mathbb{F}$, the following holds: $f(\alpha \bullet x + \beta \bullet y) = \alpha \bullet f(x) + \beta \bullet f(y)$.*

Proof :

1. Suppose f is a linear application. Then, for all $x, y \in E$ and for all $\alpha, \beta \in \mathbb{F}$
   $f(\alpha \bullet x + \beta \bullet y) = f(\alpha \bullet x) + f(\beta \bullet y)$.

2. Conversely, if f satisfies the given condition for all $x, y \in E$ and for all $\alpha, \beta \in \mathbb{F}$, then by choosing $(\alpha, \beta) = (1_{\mathbb{F}}, 1_{\mathbb{F}})$, then an arbitrary α and $\beta = 0$, we have:

$f(x+y) = f(x) + f(y) f(x+y)$ and $f(\alpha \bullet x) = \alpha \bullet f(x)$.

□

Example 0.1.3. *The function* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $f(x,y) = x + 2y$ *is a linear application.*

For all $(x_1, y_1), (x_2, y_2)$ *in* $\mathbb{R}^2$ *and for all* $\alpha, \beta \in \mathbb{R}$, *we have:*

$$\begin{aligned} f(\alpha \bullet (x_1, y_1) + \beta \bullet (x_2, y_2)) &= f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \\ &= (\alpha x_1 + \beta y_1) + 2(\alpha x_2 + \beta y_2) \\ &= \alpha(x_1 + 2y_1) + \beta(x_2 + 2y_2) \\ &= \alpha f(x_1, y_2) + \beta f(x_2, y_2). \end{aligned}$$

Example 0.1.4. *The derivative function* $D : \mathbb{F}[X] \longrightarrow \mathbb{F}[X]$ *that associates for any polynomial* P *its derivative* P' *is a linear application, since for all* $P, Q \in \mathbb{F}[X]$ *and* $\alpha, \beta \in \mathbb{F}$ *we have*

$$D(\alpha P + \beta Q) = \alpha P' + \beta Q' = \alpha D(P) + \beta D(Q).$$

D *is an endomorphism that is not an automorphism (not injective).*

Remark 0.1.5. *If* f *is a linear application, then the image of a linear combination of vectors is a linear combination of their images, i.e.,*

$$f(\sum_{i=1}^{n} \alpha_i x_i) = \alpha_i f(\sum_{i=1}^{n} x_i).$$

Where α_i *are scalars, and* x_i *are vectors.*

Image and kernel of linear mapping

Let the linear mapping $f : E \to F$.

- The image of f is denoted by the set $f(E)$ or $\text{Im}(f)$, and we write $\text{Im}(f) = \{y \in F : \exists x \in E, f(x) = y\} \subseteq F$.

- The kernel of f represents the set of elements a from E such that $f(a) = 0_F$, it's denoted as ker$f$, so ker$f = \{a \in E : f(a) = 0_F\} \subseteq E$.

Theorem 0.1.6. *Let E and F be two $\mathbb{F}$-vector spaces, and $f : E \longrightarrow F$ be a linear application. Then:*

1. $f(0_E) = 0_F$ *(0_E, 0_F are zeroes of E and F respectively).*
2. *For all $x \in E$: $f(-x) = -f(x)$.*
3. *$Imf = f(E)$ is a vector subspace of F.*
4. *$kerf = f^{-1}0_F$ is a vector subspace of E.*
5. *f is surjective if and only if $Imf = F$.*
6. *f is injective, if and only, if $kerf = \{0_E\}$.*

Example 0.1.7. *Let the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (x + 2y, x - y + 2z)$.*

1. *$kerf = \{(x, y, z) \in \mathbb{R}^3,\ (x + 2y, x - y + 2z) = (0, 0)\}$, then solving the system*

$$\begin{cases} x + 2y = 0 \\ x - y + 2z = 0. \end{cases}$$

Hence, $kerf = \{(-2y, y, \frac{3}{2}y), y \in \mathbb{R}\}$.
Hence, f is not one-one, since $kerf \neq \{0_{R^3}\}$.

2. *Finding a basis for Imf*

$$\begin{aligned} Imf &= \{f(x, y, z),\ (x, y, z) \in \mathbb{R}^3\} \\ &= \{(x + 2y, x - y + 2z), (x, y, z) \in \mathbb{R}^3\} \\ &= \{x(1, 1) + y(2, -1) + z(0, 2), x, y, z \in \mathbb{R}\}. \end{aligned}$$

Hence, Imf is the vector subspace of $\mathbb{R}^2$ spanned by the family $G = \{(1, 1), (2, -1), (0, 2)\}$, which is not a linearly independent family, otherwise $dimImf = 3$, which is impossible since $dimImf \leq dim\mathbb{R}^2$.

G contains a basis, and since set $G' = \{(1,1),(2,-1)\}$ is linearly independent, there exists a basis B of Imf such that $G' \subset B \subset G$, implying $G' = B$ and $dim_{\mathbb{R}}(Imf) = 2$, so $Imf = \mathbb{R}^2$, and consequently, f is surjective.

Theorem 0.1.8. *Let E, F and G be three $\mathbb{F}$-vector spaces, and let $f : E \to F$ and $g : F \to G$ be two linear transformations. Then $g \circ f : E \to G$ is a linear transformation.*

Proof : Let $\alpha, \beta \in \mathbb{F}$, and let $x, y \in E$, then

$$\begin{aligned} \circ f(\alpha x + \beta y) &= g(f(\alpha x + \beta y)) \\ &= g(\alpha f(x) + \beta f(y)) \\ &= \alpha g(f(x)) + \beta g(f(y)) \\ &= \alpha (g \circ f)(x) + \beta (g \circ f)(y). \end{aligned}$$

Hence, $g \circ f$ is a linear application. □