1.1.1 Linear Applications and Bases

Theorem 1.1.9. *Let E and F be two finite-dimensional $\mathbb{F}$-vector spaces, and f be a linear map from E to F.*

f is injective if and only if the image of any linearly independent subset of E is a linearly independent subset of F.

f is surjective if and only if the image of any spanning set of E is a spanning set of F.

f is an isomorphism if and only if the image of any basis of E is a basis of F.

Proof : see (TD) □

Linear Applications and Dimensions

Theorem 1.1.10. *(Fundamental theorem of linear maps) Let E and F be two finite-dimensional vector spaces over the field $\mathbb{F}$, where E be finite-dimensional spaces, and let $f : E \longrightarrow F$ be a linear application. Then:*

$$dim_{\mathbb{F}} E = dim_{\mathbb{F}}(Kerf) + dim_{\mathbb{F}}(Imf)$$

Proof : The kernel $Kerf$ is a finite-dimensional subspace. Let $k = dim_{\mathbb{F}}(kerf)$, and let $\{e_1, e_2, ..., e_k\}$ be a basis for $Kerf$, which we extend to a basis for E. Let $\{e_1, e_2, ..., e_k, e_{k+1}, ..., e_n\}$, where $n = dim_{\mathbb{F}} E$.

We need to show that $\{f(e_{k+1}), ..., f(e_n)\}$ is a basis for Imf.

For all $\alpha_{k+1}, ..., \alpha_n \in \mathbb{F}$, if $\alpha_{k+1} f(e_{k+1}) + ... + \alpha_n f(e_n) = 0$, meaning $f(\alpha_{k+1} e_{k+1} + ... + \alpha_n e_n) = 0$, then $\alpha_{k+1} e_{k+1} + ... + \alpha_n e_n \in Kerf$. Thus, we can express it as a linear combination of the basis elements of $Kerf$: $\alpha_{k+1} e_{k+1} + ... + \alpha_n e_n = \alpha_1 e1 + ... + \alpha_k e_k$, with $\alpha_1, ..., \alpha_n \in \mathbb{F}$.

By gathering all terms on the left side, we get: $-\alpha_1 e_1 - ... - \alpha_k e_k + \alpha_{k+1} e_{k+1} + ... + \alpha_n e_n = 0$. Therefore, $\alpha_1 = ... = \alpha_k = \alpha_{k+1} = ... = \alpha_n = 0$. Consequently, $\{f(e_{k+1}), ..., f(e_n)\}$ is a linearly independent set in Imf.

Let $y \in Imf$, then $y = f(x)$ for some $x \in E$ expressed as a linear combination of the basis elements of E, $x = x_1e_1 + ... + x_ke_k + x_{k+1}e_{k+1} + ... + x_ne_n$, where $x_1, ..., x_k, x_{k+1}, ..., x_n \in \mathbb{F}$. Hence, $y = x_1f(e_1) + ... + x_kf(e_k) + x_{k+1}f(e_{k+1}) + ... + x_nf(e_n) = x_{k+1}f(e_{k+1}) + ... + x_nf(e_n)$. Thus, $\{f(e_{k+1}), ..., f(e_n)\}$ is a spanning set for Imf.

In conclusion, the list $\{f(e_{k+1}), ..., f(e_n)\}$ forms a basis for Imf, and $dim_{\mathbb{F}}(Imf) = k$. Thus, $n = dim_{\mathbb{F}}(E) = dim_{\mathbb{F}}(Imf) + dim_{\mathbb{F}}(Kerf)$. □

Corollary 1.1.11. *For finite-dimensional vector spaces E and F with a linear map $f : E \longrightarrow F$, $dim_{\mathbb{F}}(Imf) = dim_{\mathbb{F}}(E)$ if and only if f is injective.*

Corollary 1.1.12. *$dim_{\mathbb{F}}(Kerf) = dimk(E) - dimk(F)$ if and only if f is surjective. Further, if E and F are finite-dimensional $\mathbb{F}$-vector spaces, then $dim_{\mathbb{F}}(E) = dim_{\mathbb{F}}(F)$ if and only if E is isomorphic to F. Additionally, $dim_{\mathbb{F}}(E) = n$ if and only if E is isomorphic to $\mathbb{F}^n$.*

Rank of a linear map

Definition 1.1.13. *We call rank of a linear application f, and we note it $rk(f)$, the dimension of Imf.*

Remark 1.1.14.

1. *The rank theorem can be written also as $rk(f) + dim(Kerf) = dim(E)$. In particular, the rank of f is always less or equal to the dimension of E.*

2. *Moreover, if F is a finite dimension space, then we have $rk(f) \leq dim(F)$, since Imf is a subspace of F.*

3. *If F is finite dimension, then f is surjective if and only if $rk(f) = dim(F)$. Since Imf is a subspace of F, and then $f = Imf$ if and only if $dim(Imf) = dim(F)$.*

Corollary 1.1.15. *Suppose that E is a finite dimension vector space, and let f be a linear application from E to F. Then*

1. *The linear application f is one to one if and only if $rk(f) = dim(E)$. Then we have $dim(E) \leq dim(F)$.*

2. *If f is onto, then F is of finite dimension and $dim(F) \leq dim(E)$.*

3. *If f is bijective, then F is finite dimension space and $dim(E) = dim(F)$.*

Exercise 1.1.16. *Let $f : R_2[X] \longrightarrow R_2[X]$ defined by*

$$f(P) = X^2P' - 2XP.$$

Prove that f is a linear application and compute its rank.

Let $\alpha, \beta \in \mathbb{R}$ and $P_1, P_2 \in R_2[X]$,

$$\begin{aligned} f(\alpha P_1 + \beta P_2) &= X^2(\alpha P_1 + \beta P_2)' - 2X(\alpha P_1 + \beta P_2) \\ &= X^2(\alpha P_1' + \beta P_2') - 2X(\alpha P_1 + \beta P_2) \\ &= \alpha(X^2P_1' - 2XP_1) + \beta(X^2P_2' - 2XP_{2)} \\ &= \alpha f(P_1) + \beta f(P_2). \end{aligned}$$

Hence, f is a linear application.
$rk(f) = dim(Imf)$. However,

$$\begin{aligned} Imf &= \{f(P),\ P \in R_2[X]\} \\ &= \{X^2(aX^2 + bX + c)' - 2X(aX^2 + bX + c),\ a, b, c \in \mathbb{R}\} \\ &= \{X^2(2aX + b) - 2X(aX^2 + bX + c),\ a, b, c \in \mathbb{R}\} \\ &= \{-bX^2 - 2cX,\ b, c \in \mathbb{R}\} \\ &= \{b(-X^2) + c(-2X),\ b, c \in \mathbb{R}\}. \end{aligned}$$

Hence, $Imf = span\{-X^2, -2X\}$, and since $\{-X^2, -2X\}$ is a linearly independent list, then it forms a basis for Imf. Then, $rk(f) = 2$.