Chapter 3

Matrices

3.1 Basic matrix notation

Definition 3.1.1. *Suppose m and n are nonnegative integers. An m-by-n matrix A is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:*

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

The notation $a_{j,k}$ denotes the entry in row j, column k of A.

Example 3.1.2. *a_{jk} equals entry in row j, column k of A*

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 9 & 6 \end{bmatrix}$$

Thus a_{23} refers to the entry in the second row, third column of A, which means that $a_{23} = 6$.

Remark 3.1.3. *1. The terms a_{ij} are called coefficients of the matrix (a_{ij}).*

2. *The matrix* (a_{ij}) *is said of type* (m, n) *if it has* m *rows and* n *columns.*

3. *The set of matrices of type* (m, n) *with coefficients in* $\mathbb{F}$*, is noted* $\mathcal{M}_{m,n}(\mathbb{F})$*.*

4. *The indexes* i, j *are respectively the number rows, and number of columns.*

5. *Two matrices* A *and* B *are equals if they are of same type and they have the same coefficients.*

$$(a_{ij}) = (b_{ij}) \ iff \ a_{ij} = b_{ij}, \ \forall i, j.$$

Example 3.1.4. $\begin{pmatrix} 1+i & 0 \\ -i & 1 \end{pmatrix}$, *where* $i^2 = -1$ *is a matrix of* $\mathcal{M}_{2,2}(\mathbb{C})$*.*

3.2 Particular matrices

1. A matrix is called null matrix and it is noted 0 if all its coefficients are zeroes, $(a_{ij}) = 0 \ iff \ a_{ij} = 0, \ \forall i, j.$

2. A matrix is called row matrix of order n if it is of type $(1, n)$.

3. A matrix is called column matrix of order n if it is of type $(n, 1)$.

4. A matrix is called square matrix of order n if it is of type (n, n).

5. A square matrix is called lower triangular if all the entries above the main diagonal are zeroes, which means $a_{ij} = 0, \ \forall i < j$. Similarly, a square matrix is called upper triangular if all the entries below the main diagonal are zeroes, which means $a_{ij} = 0, \ \forall i > j$.

6. A square matrix is called diagonal if it is upper and lower triangular.

7. The identity matrix of size n is an $n \times n$ matrix with ones on the main diagonal and zeroes elsewhere.

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

Examples 3.2.1. $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ *is a matrix of* $\mathcal{M}_{3,1}(\mathbb{R})$.

$\begin{pmatrix} 2 & 1 & 3i & 9 \end{pmatrix}$ *is a matrix of* $\mathcal{M}_{1,4}(\mathbb{C})$.

$\begin{pmatrix} 2 & i & 0 \\ 4 & 0 & -2i \end{pmatrix}$ *is a matrix of* $\mathcal{M}_{2,3}(\mathbb{C})$.

Remark 3.2.2. *The coefficients of I_n the identity matrix are noted δ_{ij}, and its called the kronecker coefficients*

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

3.2.1 Operations on matrices

3.2.1.1 Addition of matrices and scalar multiplication

Let $A = (a_{ij})$, $B = (b_{ij})$ be two matrices in $\mathcal{M}_{m,n}(F)$ and $\alpha \in \mathbb{F}$, we define $A + B$ and $\alpha.A$ by

$$A + B = (a_{ij} + b_{ij}), \quad \alpha.A = (\alpha a_{ij}).$$

Remark 3.2.3. *The sum of matrices is defined if the matrices are of same size.*

Examples 3.2.4. *Let*

$$A = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 5 & 4 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 & 2 \\ -8 & 3 & 1 & -9 \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -3 & 7 & 4 & -7 \end{pmatrix}, \quad 2.B = \begin{pmatrix} -2 & 0 & 0 & 4 \\ -16 & 6 & 2 & -18 \end{pmatrix}.$$

3.2.2 Multiplication of matrices

Let $A = (a_{ij})$ a matrix in $\mathcal{M}_{m,n}(F)$, and $B = (b_{ij})$ be a matrix in $\mathcal{M}_{n,k}(F)$, we define the product

$$A.B = (ab_{ij}) = \sum_{p=1}^{n} a_{ip} b_{pj}$$

Remark 3.2.5.

- *The product of matrices is defined if the number of columns of first matrix is the number of rows of the second matrix.*
- *The coefficients of the matrix $A.B$ are*

$$ab_{11} = \sum_{p=1}^{n} a_{1p} b_{p1} = a_{11}b_{11} + a_{12}b_{21} + ... + a_{1n}b_{n1}.$$

Similarly, the term ab_{nn} is given by

$$ab_{nn} = \sum_{p=1}^{n} a_{np} b_{pn} = a_{n1}b_{1n} + a_{n2}b_{2n} + ... + a_{nn}b_{nn}.$$

Examples 3.2.6. *Let*

$$A = \begin{pmatrix} -2 & 0 & 0 \\ -16 & 6 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

The product $A.B$ is well defined and it is given by

$$AB = \begin{pmatrix} -2 & 0 & 0 \\ -20 & 20 & 4 \end{pmatrix}.$$

However, the product BA is not defined, since $B \in \mathcal{M}_{3,3}(\mathbb{R})$ and $A \in \mathcal{M}_{2,3}(\mathbb{R})$.

3.2.2.1 Properties of matrix product

Let A, B and C three matrices, the following statements are true.

1. Non commutativity, $A.B \neq B.A$.

2. Associativity, $(A.B).C = A.(B.C)$.

3. Distributivity, $A.(B + C) = A.B + A.C$, and $(A + B).C = A.C + B.C$.

4. Identity element: $\exists I \in \mathcal{M}_n(\mathbb{F})$, $A.I = A = I.A$.

3.2.3 Transpose of a matrix

We call the transpose of a matrix $A \in \mathcal{M}_{m,n}(\mathbb{F})$, the matrix of $\mathcal{M}_{n,m}(\mathbb{F})$, noted ${}^tA$ and defined as ${}^tA = (a_{ji})$, $\forall i, j$.

Remark 3.2.7. *1. ${}^tA$ is a matrix, where its rows are columns of A.*

2. ${}^t({}^tA) = A$.

3. If ${}^tA = A$, A is called symmetric.

Examples 3.2.8. *1. If* $A = \begin{pmatrix} -1 & -1 & 0 \\ 7 & 4 & -3 \\ 9 & 1 & 8 \end{pmatrix}$, *then* ${}^tA = \begin{pmatrix} -1 & 7 & 9 \\ -1 & 4 & 1 \\ 0 & -3 & 8 \end{pmatrix}$.

2. If $A = \begin{pmatrix} -1 & -1 & 0 & 3 \\ 7 & 4 & -3 & -6 \end{pmatrix}$, *then* ${}^tA = \begin{pmatrix} -1 & 7 \\ -1 & 4 \\ 0 & -3 \\ 3 & -6 \end{pmatrix}$.

3.2.3.1 Properties of transpose

1. ${}^t(A+B) = {}^tA{}^tB$.

2. ${}^t(AB) = {}^tB{}^tA$.

3. ${}^t(\alpha A) = \alpha{}^tA$.

4. ${}^t(ABC) = {}^tC{}^tB{}^tA$.

5. If ${}^tA = -A$, A is called skew-symmetric.

6. $A{}^tA$ is always symmetric.