Exercises

Exercise 01: Let the following two sets

$$E = \{(x,y,z) \in \mathbb{R}^3,\ x+y+z=0\},\ and\ F = \{(x,y,z) \in \mathbb{R}^3,\ x-y=x+z=0\}.$$

1. Prove that E and F are subspaces of $\mathbb{R}^3$.
2. Determine a basis for E and a basis for F, then deduce their dimensions.
3. Prove that $\mathbb{R}^3 = E \oplus F$.

Exercise 02: Let

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x,y,z) \mapsto (x-y-2z, -2x+y-z, x+y+2z)$$

1. Prove that f is a linear application.
2. Determine a basis for $Kerf$, then its dimension.
3. Determine a basis for Imf, then its dimension.
4. Can you conclude that $kerf \oplus Imf = E$.
5. Determine the matrix associated to f.
6. Solve the system

$$\begin{cases} x-y-2z=3 \\ -2x+y-z=2 \\ x+y+2z=-1 \end{cases}$$

Exercise 03: Let the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

1. Compute A^2.
2. Prove that $A^2 = A + 2I_3$.
3. Deduce that A is invertible, and give its inverse A^{-1}.

Exercise:

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear application defined by:

$$f((x,y,z)) = (x-2y-z, -x+y, -x-z)$$

1. Determine the matrix A associated with the application f in the canonical basis B of $\mathbb{R}^3$, $B = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$.
2. Let $v = e_1 + e_2$, $v_2 = e_3$, $v_3 = 2e_1 + e_2 - 3e_3$ be three vectors in $\mathbb{R}^3$.
   - **(i)** Prove that $B' = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.
   - **(ii)** Determine the transition matrix P from B to B' and from B' to B. Hence, find the components of the vector $v = (1,-1,1)$ in the basis B'.
   - **(iii)** Determine the matrix A' associated with f relative to the basis B' of $\mathbb{R}^3$.