Numerical methods course

chapiter 2:

Direct methods for solving linear systems

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Numerical methods course

chapter 2:

Direct methods for solving linear systems

- **Introduction to linear system of equations**
- **Solving triangular systems**
- **Gauss Elimination Method**
- **Gaussian elimination: LU-factorization**

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- **Introduction to linear system of equations**

A linear system of m equations in n variables (unknowns) has the form

$$
\begin{aligned}
E_1 &: \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
E_2 &: \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\quad\vdots \\
E_m &: \quad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m.
\end{aligned}
$$

- where x1,x2,...,xn are the unknowns, a11,a12,... are the coefficients of the system, and b1,b2,...,bmare the constant terms.
- A linear system may behave in any one of three possible ways:
  - The system has *infinitely many solutions*.
  - The system has a single *unique solution*.
  - The system has *no solution*.

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- **Introduction to linear system of equations**

$$A\vec{x} = \vec{b}.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}.$$

- **Solving triangular systems**
  - **Definition 1;**

An upper triangular matrix $U \in R^{n\times n}$ is a matrix which all entries below the diagonal are zero: denoting the entries on A where a_{ij}, $a_{ij}=0$ pour tout i>j.

$$U = \begin{pmatrix} X & X & X & X & X \\ & X & X & X & X \\ & & X & X & X \\ & & & X & X \\ & & & & X \end{pmatrix}$$

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- **Solving triangular systems**
  - **Definition 2;**

  A Lower triangular matrix $L \in R^{n \times n}$ is a matrix which all entries above the diagonal are zero: denoting the entries on A where a_{ij}, $a_{ij}=0$ pour tout $i<j$.

$$L = \begin{pmatrix} X & & & & \\ X & X & & & \\ X & X & X & & \\ X & X & X & X & \\ X & X & X & X & X \end{pmatrix}$$

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- **Gauss Elimination Method**
  - **Aim;**

  The goal of elimination is to transform a rectangular system Ax = b, using elementary operations on A lines, into an equivalent scaled system Ux = c

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \xrightarrow{\dots \ell_{ij} \dots} \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- **Gauss Elimination Method**

For elimination without permutation of rows, we pass from A to the equivalent echelon matrix U as follows:

  - Column 1: Use the first equation to generate zeros under the first pivot.
  - Column 2: Use the new equation to generate zeros under the second pivot.
  - Column 3 to n: continue to find the n pivots of the scaled matrix U.

- **Gauss Elimination Method**

To transform any system into a triangular system, it is possible to use three elementary operations on the lines of the matrix A. This is the basis of the Gaussian elimination method. These operations are:

- **Multiplying a line by a scal** $(l_i \leftarrow \lambda l_i)$ **: Replace line i with a multiple of itself.**
- **Swapping lines** $(l_i \leftarrow l_j)$**: Swap line i and line j,**
- **The sum of the li s** $(l_i \leftarrow l_i + \lambda l_j)$ **: Replace line i with line i plus a multiple of line j.**

NB: Of the three elementary operations, only the operation $(l_i \leftarrow l_i + \lambda l_j)$ has no effect on the determinant. The permutation of two lines changes the sign, while the multiplication of a line by a scalar multiplies the determinant by this same scalar.

- **Gauss Elimination Method**
- **NB;**

The Gauss method makes it possible to calculate the determinant of the matrix A, using:

$$\det A = (-1)^p \prod_{k=1}^{n} a_{kk}^{(k)},$$

A_{kk} is the elements of the diagonal of the triangular matrix obtained using the Gaussian elimination method

- **Gauss Elimination Method**
- **Definition 3;**

The augmented matrix of the linear system is the matrix of dimension n , n +1 which we obtain by adding the right-hand side b to the matrix A, i.e.

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{array}\right)$$

- **Gauss Elimination Method**
- **step1; the triangularization of A**

$$\begin{cases} w = a_{ik}/a_{kk} \\ a_{ij}^{(i)} = aij - w.a_{kj} \end{cases}$$

$$j = k+1, \ldots, n+1, \quad i = k+1, \ldots, n \ \text{ et } \ k = 1, \ldots, n-1.$$

- **step2; the resolution of AX=B**

$$x_i = \left(b_i^{(n)} - \sum_{j=i+1}^{n} a_{ij}^{(n)} x_j \right) \Big/ a_{ii}^{(n)}.$$

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- Example 1:

$$\begin{aligned} x_1 + x_2 + 2x_3 - 3x_4 &= 1 \\ 2x_1 + 3x_2 + 6x_3 - 5x_4 &= 2 \\ 3x_1 - x_2 + 2x_3 - 7x_4 &= 5 \\ x_1 + 2x_2 + 3x_3 - x_4 &= -1 \end{aligned}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & -3 \\ 2 & 3 & 6 & -5 \\ 3 & -1 & 2 & -7 \\ 1 & 2 & 3 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -1 \end{bmatrix}$$

First, we create the **augmented matrix** =>

$$\begin{bmatrix} 1 & 1 & 2 & -3 & 1 \\ 2 & 3 & 6 & -5 & 2 \\ 3 & -1 & 2 & -7 & 5 \\ 1 & 2 & 3 & -1 & -1 \end{bmatrix}$$

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- Example 1:

step1, we procede to the triangularization of the **augmented Matrix** =>

$$\begin{bmatrix} 1 & 1 & 2 & -3 & 1 \\ 2 & 3 & 6 & -5 & 2 \\ 3 & -1 & 2 & -7 & 5 \\ 1 & 2 & 3 & -1 & -1 \end{bmatrix} \begin{matrix} L2'=L2-\frac{2}{1}L1 \\ L3'=L3-\frac{3}{1}L1 \\ L4'=L4-\frac{1}{1}L1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & -4 & -4 & 2 & 2 \\ 0 & 1 & 1 & 2 & -2 \end{bmatrix} \begin{matrix} L3''=\frac{1}{2}(L'3-\frac{-4}{1}L'2) \\ L4''=L'4-\frac{1}{1}L'2 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & -1 & 1 & -2 \end{bmatrix} \quad L4'''=2(L''4-\frac{-1}{2}L'2)$$

$$\begin{bmatrix} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 5 & -3 \end{bmatrix}$$

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- Example 1:

step2, we calculate the system solution AX=B =>

$$\begin{bmatrix} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 5 & -3 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + x_2 + 2x_3 - 3x_4 &= 1 \\ x_2 + 2x_3 + x_4 &= 0 \\ 2x_3 + 3x_4 &= 1 \\ 5x_4 &= -3. \end{aligned}$$

$x_n = b_n / a_{nn}$

$i = n-1 : -1 : 1$

$x_i = b_i$

$k = i+1 : n$

$x_i = x_i - a_{ik} x_k$

$x_i = x_i / a_{ii}$

$$x_1 = -\frac{7}{5},\ x_2 = -\frac{11}{5},\ x_3 = \frac{7}{5},\ x_4 = -\frac{3}{5}.$$

NB: Det(A) =$(-1)^0$ *(1)*(1)*(2)*(5) = 10

- Example 2:

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 2 \\ 2x_1 & + & 2x_2 & + & 5x_3 & + & 3x_4 & = & 4 \\ x_1 & + & 3x_2 & + & 3x_3 & + & 3x_4 & = & -2 \\ x_1 & + & x_2 & + & 4x_3 & + & 5x_4 & = & -2 \end{array}$$

Let's solve the system …

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- **Gaussian elimination: LU-factorization**
  - **Theory**

In this final section on matrix factorization methods for solving Ax = b we want to take a closer look at Gaussian elimination. The basic idea is to use left-multiplication of A ∈ C m×m by L (elementary) lower triangular matrices, L1, L2, . . . , Lm−1 to convert A to U upper triangular form, i.e.,

$$\underbrace{L_{m-1}L_{m-2}\ldots L_2L_1}_{=\widetilde{L}}\, A = U.$$

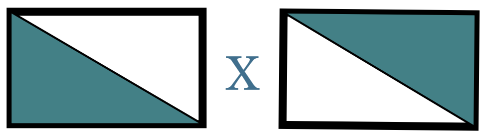
Note that the product of lower triangular matrices is a lower triangular matrix, and the inverse of a lower triangular matrix is also lower triangular. Therefore,

$$\widetilde{L}A = U \quad \Longleftrightarrow \quad A = LU,$$

- **Gaussian elimination: LU-factorization**

- To solve the system AX=B
- First, we decompose A:

A=L*U

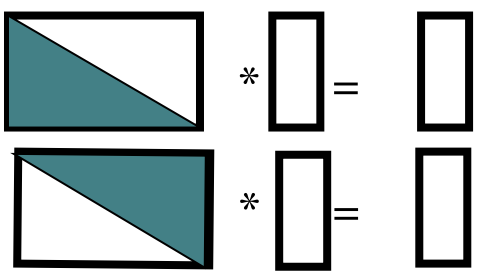


Where L is an elementary lower matrix and U is an upper matrix,

- Finaly, we calculate system solution by solving two systems:

First system ***L* y =b***

Second system ***U* x =Y***



- **Gaussian elimination: LU-factorization**

- Exemple 1:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

$$A = L * U = \begin{bmatrix} 1 & 0 & 0 \\ l_{12} & 1 & 0 \\ l_{13} & l_{32} & 1 \end{bmatrix} * \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Using Gaussian elimination on A
For i=1 to (n-1) =2
Calculate i^{th} column of L
Calculate i^{th} row of U

- **Gaussian elimination: LU-factorization**

- Exemple 1:

- For L: $i = 1 \; and \; k = 2..n, \; l_{ki} = \frac{a_{ki}^{(i-1)}}{a_{ii}^{(i-1)}}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l12 = \frac{a^{(0)}{}_{12}}{a^{(0)}{}_{11}} = \frac{4}{1} = 4 & 1 & 0 \\ l13 = \frac{a^{(0)}{}_{13}}{a^{(0)}{}_{11}} = \frac{3}{1} = 3 & l32 & 1 \end{bmatrix}$$

- For U: Eliminate Entries Below the Pivot i=1: $a_{11} = 1 \neq 0$

$$For \; j = i + 1 \ldots n, \; L_j = \; L_j - \frac{a_{ji}}{a_{ii}} \times \; L_i$$

$$A^{(0)} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{matrix} L_2 = L_2 - \frac{4}{1} L_1 \\ \\ L_3 = L_{32} - \frac{3}{1} L_1 \end{matrix} \rightarrow A^{(1)} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix}$$

- **Gaussian elimination: LU-factorization**

- Exemple 1:

2) For L: $i = 2 \; and \; k = 2..n, \;\; l_{ki} = \frac{a_{ki}^{(k-1)}}{a_{ii}^{(k-1)}}$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & l32 = \frac{a^{(1)}{}_{32}}{a^{(1)}{}_{22}} = \frac{2}{-1} = -2 & 1 \end{bmatrix}$$

Eliminate Entries Below the Pivot 2: $a_{22} = -1 \neq 0$

$$A^{(1)} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix} L_3 = L_{32} - \frac{2}{(-1)} L_1 = L_3 + 2L_1 \rightarrow A^{(2)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} = U$$

The obtained matrix is the upper triangular matrix U

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- **Gaussian elimination: LU-factorization**

- Exemple 1:

- The obtained matrix is the upper triangular matrix U

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \quad and \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$