Chapter 5 Matrix Analysis

5.1 Vector Spaces

Definition

A vector space V over a field F is a mathematical structure composed of a set of elements called vectors, which adhere to specific properties. These properties ensure that vector operations are well-defined and consistent. Below are the key properties that characterize a vector space:

- **Closure under Addition**: For any two vectors $u, v \in V$, their sum $u + v$ must also be in V. This property ensures that adding vectors together results in another vector within the same space.
- **Closure under Scalar Multiplication**: For any vector $u \in V$ and any scalar $c \in F$, the product cu must also be in V. This indicates that multiplying a vector by a scalar does not take it outside the vector space.
- **Associativity of Addition**: Vector addition is associative, meaning that for any vectors $u, v, w \in V$:

  $$(u + v) + w = u + (v + w) \qquad (57)$$

  This property ensures that the grouping of vectors during addition does not affect the outcome.

- **Commutativity of Addition**: Addition of vectors is commutative, so for any vectors $u, v \in V$:

  $$u + v = v + u \qquad (58)$$

  This means the order in which vectors are added does not change the result.

- **Identity Element of Addition**: There exists a zero vector $0 \in V$ such that for any vector $u \in V$:

  $$u + 0 = u \qquad (59)$$

  The zero vector serves as the additive identity.

- **Inverse Elements of Addition**: For every vector $u \in V$, there exists a vector $-u \in V$ such that:

  $$u + (-u) = 0 \qquad (60)$$

  This property guarantees that for every vector, there is a corresponding *"opposite"* vector that sums to the zero vector.

- **Distributive Property**: Scalar multiplication distributes over vector addition and scalar addition:

  $$c(u + v) = cu + cv$$

  $$(c + d)u = cu + du \qquad (61)$$

This means that scaling a sum of vectors is equivalent to scaling each vector and then adding.

- **Associativity of Scalar Multiplication**: Scalar multiplication is associative, so for any scalars $c, d \in F$ and any vector $u \in V$:

$$c(du) = (cd)u \tag{62}$$

This ensures that the order of scalar multiplication does not affect the result.

- **Identity Element of Scalar Multiplication**: The scalar multiplication by the identity scalar (1) yields the vector itself:

$$1u = u \tag{63}$$

This property confirms that multiplying a vector by 1 leaves it unchanged.

Example

Consider the vector space R^2 over the field of real numbers R. Elements of this vector space are vectors of the form $\begin{pmatrix} x \\ y \end{pmatrix}$where $x, y \in R$.

- **Closure under Addition**: If $u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then $u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ in R^2.
- **Closure under Scalar Multiplication**: For $c \in R$ and $u = \begin{pmatrix} x \\ y \end{pmatrix}$, the product $cu = \begin{pmatrix} cx \\ cy \end{pmatrix}$ remains in R^2.

This foundational structure provides the basis for many areas in mathematics, including linear algebra, functional analysis, and beyond.

5.2 Matrices

Definition

A matrix is a rectangular array of numbers arranged in rows and columns. It is commonly used to represent linear transformations or systems of linear equations. Each element in the matrix corresponds to a specific position defined by its row and column indices.

Notation

A matrix A of size $n \times m$ has m rows and n columns, denoted as:

$$A = \begin{pmatrix} a_{11} \, a_{12} \dots a_{1m} \\ a_{21} \, a_{22} \dots a_{2m} \\ \vdots \quad \vdots \quad \vdots \\ a_{n1} \, a_{n2} \cdots a_{nm} \end{pmatrix} \tag{64}$$

Here, a_{ij} represents the element located in the i row and j column of matrix A.

Matrices are fundamental in various mathematical fields, including linear algebra, computer science, and statistics, where they facilitate operations such as addition, multiplication, and finding determinants and eigenvalues.

5.2.1 Matrix Operations

5.2.1.1 Addition

Definition

Two matrices $A \in R_{nxm}$ and $B \in R_{nxm}$ can be added if they have the same dimensions: The addition of two matrices A and B results in a new matrix C, where each element c_{ij} is calculated by adding the corresponding elements of A and B:

$$C = A + B = \left[c_{ij} = a_{ij} + b_{ij}\right]_{n\times m}$$

$$= \begin{pmatrix} a_{11} + b_{11}\, a_{12} + b_{12} \ldots\, a_{1m} + b_{1m} \\ a_{21} + b_{21}\, a_{22} + b_{22} \ldots\, a_{2m} + b_{2m} \\ \vdots \qquad \vdots \qquad \vdots \\ a_{n1} + b_{n1} a_{n2} + b_{n2} \cdots\, a_{nm} + b_{nm} \end{pmatrix} \quad (65)$$

- **Example:**

1) If we have: $A = \begin{pmatrix} 1 & -3 \\ -2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 3 \\ 7 & 2 \end{pmatrix}$

Then the sum $C = A + B$ is:

$$C = A + B = \begin{pmatrix} 1-5 & -3+3 \\ -2+7 & -1+2 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 5 & 1 \end{pmatrix}$$

2) If we have $A = \begin{pmatrix} 1 & -3 & 0 \\ 0 & -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 7 \\ 1 & 0 \end{pmatrix}$

Matrix A is of size 2×3 and matrix B is of size 2×2. Since their dimensions do not match, we cannot add them directly.

5.2.1.2 Scalar Multiplication

Definition

Scalar multiplication involves multiplying each element of a matrix A by a scalar c.

If A is a matrix of size $n \times m$ given by:

$$A = \begin{pmatrix} a_{11}\, a_{12} \ldots\, a_{1m} \\ a_{21}\, a_{22} \ldots\, a_{2m} \\ \vdots \quad \vdots \quad \vdots \\ a_{n1} a_{n2} \cdots\, a_{nm} \end{pmatrix} \quad (66)$$

then the result of multiplying A by the scalar$c$ is another matrix cAdefined as:

$$c.A = \begin{pmatrix} c.a_{11} & c.a_{12} \dots & c.a_{1m} \\ c.a_{21} & c.a_{22} \dots & c.a_{2m} \\ \vdots & \vdots & \vdots \\ c.a_{n1} & c.a_{n2} \cdots & c.a_{nm} \end{pmatrix} \tag{67}$$

Notation

Properties of Scalar Multiplication:

1) Distributive Property: $c(A + B) = cA + cB$
2) Associative Property: $c(dA) = (cd)A$
3) Identity Element: $1A = A$

Example

Let $A = \begin{pmatrix} 1 & -5 \\ 0 & -1 \end{pmatrix}$ and let $c = 3$.

Then,

$$cA = 3\begin{pmatrix} 1 & -5 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \times 1 & 3 \times (-5) \\ 3 \times 0 & 3 \times (-1) \end{pmatrix} = \begin{pmatrix} 3 & -15 \\ 0 & -3 \end{pmatrix} \quad .$$

This demonstrates how each element of the matrix A is multiplied by the scalar c.

5.2.1.3 Matrix Multiplication

Definition

The product of two matrices AB is defined when the number of columns in matrix A is equal to the number of rows in matrix B. Specifically, if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the resulting product AB will be an $m \times p$ matrix.

Notation

- Let A be an $m \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix} \tag{68}$$

- Let B be an $n \times p$ matrix:

$$B = \begin{pmatrix} b_{11} & b_{12} \dots & b_{1p} \\ b_{21} & b_{22} \dots & b_{2p} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} \cdots & b_{np} \end{pmatrix} \tag{69}$$

Calculation of the Product AB

The product AB is calculated by taking the dot product of the rows of A with the columns of B. The element in the i^{th} row and j^{th} column of the resulting matrix $C = AB$ is given by:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad (68)$$

where:

- c_{ij} is the entry in the i^{th} row and j^{th}column of the matrix C.
- a_{ik} is the entry from the i^{th} row of A.
- b_{kj} is the entry from the j^{th} column of B.

Notation

Properties of Matrix Multiplication

1) Non-Commutativity: In general, $AB \neq BA$.
2) Associativity: $(AB)C = A(BC)$.
3) Distributive: $A(B + C) = AB + AC$.

Example

Let:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \ \ and\ B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

The product AB is calculated as follows: $AB = \begin{pmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$

Matrix multiplication is a fundamental operation in linear algebra, with wide-ranging applications in various fields including computer science, engineering, and economics. Understanding the conditions for multiplication, the calculation method, and properties is essential for further exploration of linear transformations and systems of equations.

5.2.2 Relationships between Linear Mappings and Matrices

Definition

Every linear mapping $T: R^n \rightarrow R^m$ can be represented by a matrix A such that:

$$T(x) = Ax \quad (69)$$

for all $x \in R^n$. Here, A is an $m \times n$ matrix where the action of T on a vector x corresponds to the matrix multiplication of A and x.

5.2.2.1 Representation of the Matrix

If T is defined by its effect on the standard basis vectors $e_1, e_2, \ldots, e_n$ of R^n the columns of the matrix A are given by:

$$A = (T(e_1)\ T(e_2)\ \ldots\ T(e_n)) \tag{70}$$

where each $T(e_2)$ is an m-dimensional vector.

Notation

The properties are :

1) Linearity: The matrix A captures the linearity of the mapping T, meaning: $T(c_1x_1 + c_2x_2) = c_1T(x_1) + c_2T(x_2)$ for any scalars c_1, c_2 and vectors $x_1, x_2 \in R^n$.

2) Composition: If $T_1: R^n \to R^m$ and $T_2: R^n \to R^m$ are linear mappings with matrices A_1 and A_2, respectively, then the composition $T_2 \circ T_1$ can be represented by the matrix product A_2A_1.

Understanding the relationship between linear mappings and matrices allows for the translation of abstract linear transformations into concrete matrix operations, facilitating analysis and computation in various mathematical and applied contexts.

Example

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Calculation: $T(x) = Ax = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

5.2.3 Inverse of a Matrix

The inverse A^{-1} of a matrix A satisfies the following conditions:

$$AA^{-1} = I \text{ and } A^{-1}A = I \tag{71}$$

$$A^{-1} = \frac{1}{\det(A)} \times adj(A)$$

where I is the identity matrix and $adj(A) is$ Adjugate.

The adjugate (or adjoint) of a matrix A, is the transpose of the cofactor matrix of A. It is calculated by:

- Finding the cofactor for each element of A.
- Transposing the resulting matrix of cofactors.

Example

For

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$$

Finding A^{-1} involves calculating the determinant and the adjugate. we will follow these steps:

1. Calculate the Determinant of A.
2. Find the Adjugate of A.
3. Use the formula for the inverse.

$$\det(A) = 1\begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix} = -24 + 40 - 15 = 1$$

Calculate the Cofactor Matrix

To find the cofactor C_{ij}, we calculate the determinant of the 2×2 matrix obtained by deleting the i row and j column and apply the sign based on the position.

$$C_{ij} = (-1)^{i+j} det(i,j)$$

1. **Cofactor** C_{11} (delete row 1, column 1):

$$C_{11} = det\begin{pmatrix} 1 & 4 \\ 6 & 0 \end{pmatrix} = -24$$

2. **Cofactor** C_{12} (delete row 1, column 2):

$$C_{12} = -det\begin{pmatrix} 0 & 4 \\ 5 & 0 \end{pmatrix} = 20$$

3. **Cofactor** C_{13} (delete row 1, column 3):

$$C_{13} = det\begin{pmatrix} 0 & 1 \\ 5 & 6 \end{pmatrix} = -16$$

4. **Cofactor** C_{21} (delete row 2, column 1):

$$C_{21} = -det\begin{pmatrix} 2 & 3 \\ 6 & 0 \end{pmatrix} = 18$$

5. **Cofactor** C_{22} (delete row 2, column 2):

$$C_{22} = det\begin{pmatrix} 1 & 4 \\ 6 & 0 \end{pmatrix} = -15.$$

6. **Cofactor** C_{23} (delete row 2, column 3):

$$C_{23} = -det\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix} = 4$$

7. **Cofactor** C_{31} (delete row 3, column 1):

$$C_{31} = det\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = 5$$

8. **Cofactor** C_{32} (delete row 3, column 2):

$$C_{32} = -det\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} = -4$$

9. **Cofactor** C_{33} (delete row 3, column 3):

$$C_{33} = -det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = 1$$

. Putting it all together, the cofactor matrix , and Taking the transpose of the cofactor matrix :

$$C = \begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{pmatrix} \rightarrow \boldsymbol{adj(A) = C^T} = \begin{pmatrix} \mathbf{-24} & \mathbf{18} & \mathbf{5} \\ \mathbf{20} & \mathbf{-15} & \mathbf{-4} \\ \mathbf{-5} & \mathbf{4} & \mathbf{1} \end{pmatrix}$$

Using the formula for the inverse:

$A^{-1} = \frac{1}{\det(A)} \times adj(A)$

Since $det(A) = 1$, The inverse of the matrix A is:

$$A^{-1} = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

5.2.4 Trace and Determinant of a Matrix

Definition 1

The trace of a square matrix A, denoted $tr(A)$, is defined as the sum of the diagonal elements:

$$\boldsymbol{tr(A) = a_{11} + a_{22} + \cdots + a_{nn}} \qquad (72)$$

Properties of the Trace

1) Linearity: $tr(A + B) = tr(A) + tr(B)$ for any two square matrices A and B of the same size.
2) Scalar Multiplication**:** $tr(cA) = ctr(A)$ for any scalar c.
3) Transpose $tr(A^T) = tr(A)$.

Definition 2

The determinant of a square matrix A, denoted $det(A)$, is a scalar value that provides important properties regarding the matrix, including whether it is invertible.

Properties of the Determinant

1) Multiplicative: $det(AB) = det(A).det(B)$ for any two square matrices A and B.
2) Invertibility: A matrix A is invertible if and only if $\det(A) \neq 0$.
3) Effect of Row Operations:

- Swapping two rows multiplies the determinant by -1.
- Multiplying a row by a scalar c multiplies the determinant by C.
- Adding a multiple of one row to another does not change the determinant.

5.2.5 Eigenvalues and Eigenvectors

Definition

For a square matrix A, a non-zero vector v is called an eigenvector and the corresponding scalar λ is called an eigenvalue if *(see chapter 4*):

$$Av = \lambda v \tag{73}$$

5.2.5.1 Finding Eigenvalues

To find the eigenvalues of A, solve the characteristic equation:

$$det(A - \lambda I) = 0 \tag{74}$$

where I is the identity matrix of the same size as A.

5.2.5.2 Finding Eigenvectors

Once the eigenvalues are determined, substitute each eigenvalue λ back into the equation $(A - \lambda I)v = 0$ to find the corresponding eigenvectors.

Properties

1) Sum of Eigenvalues: The sum of the eigenvalues of A equals the trace of A.
2) Product of Eigenvalues**:** The product of the eigenvalues equals the determinant of A.

5.2.6 Similar Matrices

Definition

Two square matrices A and B are said to be similar if there exists an invertible matrix P such that:

$$B = P^{-1}AP \tag{75}$$

Properties of Similar Matrices

1) Same Eigenvalues**:** Similar matrices share the same eigenvalues.
2) Same Determinant and Trace: If A and B are similar, then $det(A) = det(B)$ and $tr(A) = tr(B)$).
3) Invariant under Similarity**:** Many properties of matrices, such as rank and characteristic polynomial, are invariant under similarity.

5.2.7 Some Special Matrices

Definition 1

A diagonal matrix is a square matrix in which all elements outside the main diagonal are zero. It can be represented as:

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & d_n \end{pmatrix}$$

where $d_1, d_2, .. d_n$ are the diagonal entries.

Properties

1) **Eigenvalues:** The eigenvalues of a diagonal matrix are simply its diagonal entries:

   Eigenvalues=$d_1, d_2, .. d_n$

2) **Determinant:** The determinant of a diagonal matrix is the product of its diagonal entries:

   $\det(A) = d_1 \times d_2 \times, . \times d_n$

3) **Inverse:** If $d_i \neq 0$ for all i, the inverse of a diagonal matrix D is also a diagonal matrix:

$$D^{-1} = \begin{pmatrix} 1/d_1 & 0 & \dots & 0 \\ 0 & 1/d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1/d_n \end{pmatrix}$$

Definition 2

A matrix Q is orthogonal if its transpose is equal to its inverse:

$$Q^T Q = I$$

Properties

1) Orthonormal Vectors: The columns (and rows) of Q are orthonormal vectors. This means:

   - Each column vector has a length of 1.
   - Any two different columns are orthogonal to each other.

2) **Determinant:** The determinant of an orthogonal matrix is either +1 or −1:

$$det(Q) = \pm 1$$

3) **Inverse:** The inverse of an orthogonal matrix is its transpose:

$$Q^{-1} = Q$$

Definition 3

A matrix A is **symmetric** if it is equal to its transpose:

$$A = A^T$$

Properties

1) Real Eigenvalues: All eigenvalues of a symmetric matrix are real.
2) Diagonalization: There exists an orthogonal matrix Q such that:

$$A = Q\Lambda Q^T$$

where Λ is a diagonal matrix containing the eigenvalues of A.

3) Quadratic Form: For any vector x: $x^T A\ x$ is a vreal number.

5.3 Norms and Inner Products

Norms and inner products are fundamental concepts in linear algebra that provide a way to measure the size of vectors and the angle between them. These tools are essential for various applications in mathematics, physics, and engineering.

5.3.1 Definitions

Vector Norms

A norm is a function that assigns a non-negative length or size to vectors in a vector space. It is denoted as $\| x \|$\for a vector x.

Properties of Norms

For any vector x and scalar c:

1) Non-negativity: $\| x \| \geq 0$ and $\| x \| = 0$ if and only if $x = 0$.
2) Scalar multiplication: $\| cx \| = | c | \| x \|$.
3) Triangle inequality: $\| x + y \| \leq \| x \| + \| y \|$.

Common Norms

1. **1-Norm (Manhattan Norm)**: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$
2. **2-Norm (Euclidean Norm)**: $\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$

3. **Infinity Norm**: $\|x\|_\infty = \max_i |x_i|$

5.3.2 Inner Products and Vector Norms

Inner Product

An inner product is a generalization of the dot product that allows us to define angles and lengths in a vector space. For two vectors $x, y \in R^n$, the inner product is denoted as $\langle x, y \rangle$.

Properties of Inner Products

1) Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2) Linearity**:** $\langle cx + z, y \rangle = c\langle x, y \rangle + \langle z, y \rangle$
3) Positive definiteness**:** $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if x=0.

Relationship Between Norms and Inner Products

The norm of a vector can be derived from the inner product:

$$\|x\| = \sqrt{(x, x)}$$

Examples of Inner Products

1) Standard Inner Product: $\langle x, y \rangle = \sum_{i=1}^{n} x_i \, y_i$
2) Weighted Inner Product**:** $\langle x, y \rangle = \sum_{i=1}^{n} w_i x_i \, y_i$ where wi>0w_i > 0wi>0 are weights.

5.3.3 Matrix Norms

Definition

A matrix norm is a function that assigns a non-negative size to matrices, analogous to vector norms. It is denoted as $\| A \|$ for a matrix A.

Properties of Matrix Norms

For any matrices A and B of appropriate dimensions and scalar c:

1) Non-negativity: $\| A \| \geq 0$ and $\| A \| = 0$ if and only if A is the zero matrix.
2) Scalar multiplication**:** $\| cA \| = | c | \| A \| |$.
3) Triangle inequality: $\| A + B \| \leq \| A \| + \| B \|$.

Common Matrix Norms

1. **Frobenius Norm**:

$$\| A \|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{tr(A^* A)}$$

where $A^* *$ is the conjugate transpose of A.

2. **1-Norm**:

$$\| A \|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$$

This norm is the maximum absolute column sum of the matrix.

3. **Infinity Norm**:

$$\| A \|_\infty = \max_{1 \leq j \leq m} \sum_{i=1}^{n} |a_{ij}|$$

This norm is the maximum absolute row sum of the matrix.

4. **2-Norm (Spectral Norm)**:

$$\| A \|_2 = \sigma_{max}$$

where σ_{max} is the largest singular value of A.

Applications of Norms and Inner Products

- **Stability Analysis**: In control theory, norms are used to analyze the stability of systems.
- **Optimization**: In machine learning, norms help in regularization techniques to prevent overfitting.
- **Numerical Analysis**: Norms measure error and convergence rates of numerical methods.

QCM

Here are multiple-choice questions (MCQs) based on previous section:

Question 1:

1. Which of the following is NOT a property of a vector space?
A) Closure under addition
B) Existence of a zero vector
C) Commutativity of multiplication
D) Closure under scalar multiplication
Answer: C) Commutativity of multiplication

Question 2:

What is the result of the matrix multiplication $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$?.

A) $\begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$
B) $\begin{pmatrix} 31 & 52 \\ 15 & 24 \end{pmatrix}$
C) $\begin{pmatrix} 14 & 12 \\ 11 & 3 \end{pmatrix}$
D) $\begin{pmatrix} 26 & 7 \\ 19 & 32 \end{pmatrix}$
Answer: $\begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$.

Question 3:

If A is a matrix representing a linear mapping, which of the following statements is true?
A) The matrix has the same dimensions as the vector space it maps to.
B) The rank of the matrix is always equal to its dimension.
C) Every linear mapping can be represented by a matrix.
D) The inverse of a matrix always represents a linear mapping.
Answer: C) Every linear mapping can be represented by a matrix.

Question 4:

What is the determinant of the matrix $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$?

A) -2.
B) 2.
C) 4.
D) 1.
Answer: A) -2.

Question 5:

Which of the following is a property of inner products in vector spaces?
A) It is commutative but not associative.
B) It can produce a scalar result.
C) It requires at least three vectors.
D) It can be negative.
Answer: B) It can produce a scalar result.