3.4 Trace of matrix

Definition 3.4.1. *The sum of the diagonal elements of a square matrix A is called the trace of A, denoted by $Tr(A)$.*

Example 3.4.2. *Let* $A = \begin{pmatrix} 4 & -1 & 3 \\ 7 & 1 & 2 \\ 9 & 0 & -7 \end{pmatrix}$, *its trace is* $Tr(A) = -2$.

Theorem 3.4.3. *Let A and B be two matrices of order n, then*

1. $Tr(A + B) = Tr(A) + Tr(B)$.
2. $\forall \alpha \in \mathbb{F},\ Tr(\alpha.A) = \alpha.Tr(A)$.
3. $Tr({}^tA) = Tr(A)$.
4. $Tr(AB) = Tr(BA)$.

3.5 Determinants

3.5.1 Determine of 2×2 and 3×3 matrices

1. Given 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

   We define the determinant of A as:

   $$det(A) = det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

   Example 3.5.1. *Compute determinant of A such that*
   $A = \begin{pmatrix} 2 & -1 \\ 7 & 3 \end{pmatrix}$, $det(A) = 2(3) - 7(-1) = 13$.

2. Given 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

We define the determinant of A as:

$$det(A) = det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 3.5.2. *Compute determinant of A such that*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 1 & -4 & 1 \end{pmatrix},\ det(A) = 1((3 \times 1) - (-4 \times 1)) = 7.$$

Definition 3.5.3. *Let A be a 3×3 matrix, let (a_{jk}) be 2×2 matrix obtained from A by deleting the j^{th} row and k^{th} column. Defining the co-factor of a_{jk} to be the number $C_{jk} = (-1)^{j+k}det(a_{jk})$. Define the determinant to be*

$$det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

This definition is called the expansion of the determinant along the 1^{st} row.

Remark 3.5.4. *A helpful way to remember the sign of a co-factor is to use the matrix*

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

Example 3.5.5. *Compute the determinant of A, where*

$$A = \begin{pmatrix} 4 & -2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 6 \end{pmatrix}.$$

$det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 77.$

3.5.2 Determine of $n \times n$ matrices

We define the determinant of a general $n \times n$ matrix as follows.

Let A be a $n \times n$ matrix, let (a_{jk}) be the $(n-1) \times (n-1)$ matrix obtained from A by deleting the j^{th} row and k^{th} column, and let $C_{jk} = (-1)^{j+k}det(a_{jk})$ be the (j,k)-cofactor of A. The determinant of A is defined to be:

$$det(A) = a_{11}C_{11} + a_{12}C_{12} + ... + a_{1n}C_{1n}.$$

Theorem 3.5.6. *Let A be a $n \times n$ matrix, then $det(A)$ may be obtained by a cofactor expansion along any row or column of A:*

$$det(A) = a_{j1}C_{j1} + a_{j2}C_{j2} + ... + a_{jn}C_{jn}.$$

Corollary 3.5.7. *If A has a row or column containing all zeroes then $det(A) = 0$.*

Corollary 3.5.8. *For any square matrix A, it holds that $det(A) = det({}^tA)$.*

Proof : Expanding along j^{th} row of A is equivalent to expanding along j^{th} column of ${}^tA$. □

Example 3.5.9. *Compute the det of A:*

$$A = \begin{pmatrix} 4 & 0 & 3 & -1 \\ 2 & 3 & 5 & 0 \\ 1 & 0 & 6 & 1 \\ 1 & 0 & 2 & 0 \end{pmatrix}.$$

Expanding along the second column, we find

$$det(A) = 3det \begin{pmatrix} 4 & 3 & -1 \\ 1 & 6 & 1 \\ 1 & 2 & 0 \end{pmatrix} = -3$$

Corollary 3.5.10. *If A has two rows (or two columns) that are equal, then $det(A) = 0$.*

Theorem 3.5.11. *Let $A, B \in \mathcal{M}_n(\mathbb{R})$, then $det(AB) = det(A)det(B)$.*

Corollary 3.5.12. *For any square matrix $det(A^k) = (det(A))^k$.*

Theorem 3.5.13. *Let $A \in \mathcal{M}_n(\mathbb{F})$, $B = \beta A$ that is B is obtained by multiplying every entry of A by β, then $det(B) = \beta^n det(A)$.*

3.5.3 The Cofactor matrix

Recall that $det(A) = a_{j1}C_{j1} + a_{j2}C_{j2} + ... + a_{jn}C_{jn}$, where $C_{jk} = (-1)^{j+k}det(a_{jk})$ is called the (j, k)-cofactor of A, and

$$a_j = [a_{j1} \ a_{j2} \ \ldots \ a_{jn}]$$

is the j^{th} row of A. If $C_j = [C_{j1}\ C_{j2}\ ...\ C_{jn}]$ then

$$det(A) = [a_{j1}\ a_{j2}\ \dots\ a_{jn}] \begin{bmatrix} C_{j1} \\ C_{j2} \\ . \\ . \\ C_{jn} \end{bmatrix} = a_j C_j^t.$$

On the other hand, if $j \neq k$ then

$$a_j C_j^t = \begin{cases} det(A), & if\ j = k, \\ 0, & if\ j \neq k. \end{cases}$$

From the co-factor matrix, we can write $A(\frac{1}{det(A)})(cof(A))^t = I_n$. Hence, we deduce $A^{-1} = \frac{1}{det(A)}{}^t cof(A)$.

The co-factor method is an alternative method to find the inverse of an invertible matrix. Recall that for any matrix $A \in \mathbb{R}^{n\times n}$, if we expand along the j^{th} row.

Suppose that B is the matrix obtained from A by replacing row a_{ij} with a distinct row a_k. To compute $det(B)$ expand along the j^{th} row, $b_j = a_k$, $det(B) = a_k C_j^t = 0$.

Theorem 3.5.14. *The determinant of triangular matrix is the product of its diagonal entries.*

Theorem 3.5.15. *Suppose that $A \in \mathcal{M}_{n,n}(\mathbb{R})$ and let B be the matrix obtained by interchanging two rows of A. Then, $det(B) = -det(A)$.*

3.6 Invertibility of matrices

Theorem 3.6.1. *A square matrix A is invertible if and only if $det(A) \neq 0$.*

Corollary 3.6.2. *Let A be an invertible matrix, then $det(A^{-1}) = \frac{1}{det(A)}$.*

Proof : We have $AA^{-1} = A^{-1}A = I_n$, then $det(AA^{-1}) = det(I_n)$. Which implies $det(A)det(A^{-1}) = det(I_n) = 1$. Hence, $det(A^{-1}) = \frac{1}{det(A)}$. □

3.6.1 The cofactor method

Define the cofactor matrix as follows

$$Cof(A) = \begin{bmatrix} C_{11} & \dots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \dots & C_{nn} \end{bmatrix}.$$

We have $A.Cof(A)^t = det(A).I_n$, this leads to the following formula for the inverse

$$A^{-1} = \frac{1}{det(A)} Cof(A)^t.$$

The inverse of 2×2 matrix, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. with $a_{11}a_{22} - a_{12}a_{21} \neq 0$, the cofactor matrix of A is given by

$$Cof(A) = \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}.$$

Hence, $A^{-1} = \frac{1}{det(A)} Cof(A)^t = \frac{1}{det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

Exercise 3.6.3. *Compute the inverse of the following invertible matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 4 & 2 & 1 \end{pmatrix}.$$

3.6.2 Gauss Jordan method

To find the inverse of a matrix using the Gauss-Jordan method, we start by augmenting the matrix with the identity matrix, then perform row operations to transform the original matrix to the identity matrix.

Example 3.6.4. *Compute the inverse of the matrix A, where*

$$A = \begin{pmatrix} -3 & 2 & 2 \\ 6 & -6 & 4 \\ 3 & -4 & 7 \end{pmatrix}.$$

We have $AA^{-1} = I_3$*. Let*

$$\begin{array}{c} L1 \\ L2 \\ L3 \end{array}\left(\begin{array}{ccc|ccc} -3 & 2 & 2 & 1 & 0 & 0 \\ 6 & -6 & 4 & 0 & 1 & 0 \\ 3 & -4 & 7 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_2 \to L_2 + 2L_1} \left(\begin{array}{ccc|ccc} -3 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 8 & 2 & 1 & 0 \\ 3 & -4 & 7 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_3 \to L_3 + L_1} \left(\begin{array}{ccc|ccc} -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 8 & 2 & 1 & 0 \\ 0 & -2 & 9 & 1 & 0 & 1 \end{array}\right) \xrightarrow{L_3 \to L_3 - L_2} \left(\begin{array}{ccc|ccc} -3 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 8 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{L_1 \to L_1 + L_2} \left(\begin{array}{ccc|ccc} -3 & 0 & 10 & 3 & 1 & 0 \\ 0 & -2 & 8 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow{L_1 \to L_1 - 10L_3 \wedge L_2 \to L_2 - 8L_3} \left(\begin{array}{ccc|ccc} -3 & 0 & 0 & 13 & 11 & -10 \\ 0 & -2 & 0 & 10 & 9 & -8 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Therefore,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-13}{3} & \frac{-11}{3} & \frac{10}{3} \\ 0 & 1 & 0 & \frac{-10}{2} & \frac{-9}{2} & \frac{8}{2} \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Hence,

$$A^{-1} = \begin{pmatrix} \frac{-13}{3} & \frac{-11}{3} & \frac{10}{3} \\ \frac{-10}{2} & \frac{-9}{2} & \frac{8}{2} \\ -1 & -1 & 1 \end{pmatrix}.$$

3.7 Rank of a matrix

Definition 3.7.1. *The rank of a matrix* A *is the dimension of the vector subspace generated (spanned) by its columns. This correspond to the maximal number of linearly independents columns of* A*. This in turn, is identical to the dimension of the vector subspace spanned by its rows.*

Example 3.7.2. *The matrix*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix}$$

has rank 2*, since* $\{v_1, v_{2,v_3}\}$ *are linearly dependents. However,* $\{v_1, v_2\}$ *are linearly independents.*

3.8 Properties of inverse matrix

1. Let A be an invertible matrix, then $(A^{-1})^{-1} = A$.

2. Let A and B be invertible matrices, the $(AB)^{-1} = B^{-1}A^{-1}$. More general, if $A_1, A_2..., A_n$ are invertible matrices, then

$$(A_1.A_2...A_n)^{-1} = A_n^{-1}.A_{n-1}^{-1}....A_2^{-1}.A_1^{-1}.$$

3. If $det(A) = 0$, then A is called a singular matrix.

4. If a non-singular square matrix A is symmetric, then A^{-1} is also symmetric.

5. If A be an invertible matrix, then $AA^{-1} = A^{-1}A = I$.

3.9 Matrix of a linear mapping

3.9.1 Matrix representation of a linear mapping

Let $f : E \to F$ be a linear mapping, with E and F are finite dimensional vector spaces over a field $\mathbb{F}$, with dimensions n and m respectively. Let $B = \{e_1, e_2, ..., e_n\}$ be a basis for E, and $B' = \{e'_1, e'_2, ..., e'_m\}$ be a basis for F.

Definition 3.9.1. *Since $B' = \{e'_1, e'_2, ..., e'_m\}$ is a basis for F, then there exists unique scalars $a_{ij} \in \mathbb{F}$ such that*

$$f(e_j) = a_{1j}e'_1 + ... + a_{nj}e'_m, \ for\ 1 \leq j \leq n.$$

We can collect these scalars in an $n \times m$ matrix as follows:

$$M(f) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.$$

Remark 3.9.2. *Note that $M(f)$ depends on the linear mapping and the choice of bases.*

Examples 3.9.3. *1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping such that $f(x, y) = (x+y, 2x-y)$, with respect to the canonical basis of $\mathbb{R}^2$, $B = \{e_1(1, 0), e_2(0, 1)\}$, we get $f(1, 0) = (1, 2)$ and $f(0, 1) = (1, -1)$. Then, the corresponding matrix is*

$$M(f) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

2. *Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be a linear mapping such that $f(x, y, z, t) = (2x - 3y + z, x - y + z - 2t)$, let $B = \{e_1(1, 0, 0, 0), e_2(0, 1, 0, 0), e_3(0, 0, 1, 0), e_4(0, 0, 0, 1)\}$ be the canonical basis of $\mathbb{R}^4$, we have*

$$f(e_1) = (2, 1),\ f(e_2) = (-3, -1),\ f(e_3) = (1, 1),\ \text{and } f(e_4) = (0, -2).$$

Therefore, the corresponding matrix is

$$M(f) = \begin{pmatrix} 2 & -3 & 1 & 0 \\ 1 & -1 & 1 & -2 \end{pmatrix}.$$

Proposition 3.9.4. *Let E and F are finite dimensional vector spaces over a field $\mathbb{F}$, with dimensions n and m respectively, let $\{e_i\}_{1 \leq i \leq n}$, and $\{u_j\}_{1 \leq j \leq m}$ bases of E and F respectively. Then, the application $T : \mathcal{L}(E, F) \to \mathcal{M}_{m,n}(\mathbb{F})$ is an isomorphism of vector spaces. Which means, $M(f + g) = M(f) + M(g)$, $M(\lambda f) = \lambda M(f)$, where $\lambda \in \mathbb{F}$,.*

Proposition 3.9.5. *Let E, F and G are finite dimensional vector spaces over a field $\mathbb{F}$, with dimensions n, m and k respectively, let $\{e_i\}_{1 \leq i \leq n}$, $\{u_j\}_{1 \leq j \leq m}$ and $\{v_l\}_{1 \leq l \leq k}$ bases of E, F and G respectively. Let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$, then we have $M(g \circ f) = M(g) \times M(f)$.*

Proposition 3.9.6. *Let E and F be two vector spaces over a field $\mathbb{F}$ with same dimension n. Let $\{e_i\}_{1 \leq i \leq n}$, and $\{u_j\}_{1 \leq j \leq n}$ bases of E and F respectively. A linear application $f \in \mathcal{L}(E, F)$ is bijective if and only if $M(f)$ is invertible. Moreover, $M(f^{-1}) = (M(f))^{-1}$.*

3.9.2 Transition matrix

Let E be a vector-space of dimension n, and $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ two bases of E.

Definition 3.9.7. *We call transition matrix from basis $\{e_1, \ldots, e_n\}$ to the basis $\{e'_1, \ldots, e'_n\}$, the matrix noted $P_{\{e_1,\ldots,e_n\} \longrightarrow \{e'_1,\ldots,e'_n\}}$, where the columns are the coordinates of vectors $\{e'_1, \ldots, e'_n\}$ in the basis $\{e_1, \ldots, e_n\}$.*
The matrix $P_{\{e_1,\ldots,e_n\} \longrightarrow \{e'_1,\ldots,e'_n\}}$ is the matrix of the identity Id_E in the basis $\{e_i\}_{1 \leq i \leq n}$ and $\{e'_i\}_{1 \leq i \leq n}$.

Proposition 3.9.8. *The transition matrix* $P_{\{e_1,\dots,e_n\}\longrightarrow\{e'_1,\dots,e'_n\}}$ *is invertible and its inverse is the transition matrix* $P_{\{e'_1,\dots,e'_n\}\longrightarrow\{e_1,\dots,e_n\}}$.
Let $x \in E$ *of coordinates* $(x_1, x_2, \dots, x_n)$ *in the basis* $\{e_1, \dots, e_n\}$ *and of coordinates* $(x'_1, x'_2, \dots, x'_n)$ *in the basis* $\{e'_1, \dots, e'_n\}$.

We note P *the transition matrix from* $\{e_i\}_{1\le i\le n}$ *to* $\{e'_i\}_{1\le i\le n}$ *and* $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $X' = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$, *we have then* $X' = P^{-1}X$.

Examples 3.9.9. *Let* $B = \{(1, 1, 1), (1, -1, 1), (0, 0, 1)\}$ *and* $B' = \{(2, 2, 0), (0, 1, 1), (1, 0, 1)\}$.

1. *Find the transition matrix from* B *to* B'.

2. *Find the transition matrix from* B' *to* B.

3.9.3 Change of basis

Proposition 3.9.10. *Let* $f \in \mathcal{L}(E, F)$, *and let* $\{e_1, \dots, e_n\}$ *and* $\{e'_1, \dots, e'_n\}$ *two bases of* E, *and* $\{u_1, \dots, u_p\}$ *and* $\{u'_1, \dots, u'_p\}$ *two bases of* F. *We note* $A = M(f)_{e_i,u_i}$, $B = M(f)_{e'_i,u'_i}$, $P = P_{\{e_1,\dots,e_n\}\longrightarrow\{e'_1,\dots,e'_n\}}$ *and* $Q = P_{\{u_1,\dots,u_p\}\longrightarrow\{u'_1,\dots,u'_p\}}$. *Then, we have* $B = Q^{-1}AP$.

Corollary 3.9.11. *Let* $f \in \mathcal{L}(E, E)$, *and let* $\{e_1, \dots, e_n\}$ *and* $\{e'_1, \dots, e'_n\}$ *two bases of* E, *note* $A = M(f)_{e_i,u_i}$, $B = M(f)_{e'_i,u'_i}$ *and* $P = P_{\{e_1,\dots,e_n\}\longrightarrow\{e'_1,\dots,e'_n\}}$. *Then,* $B = P^{-1}AP$.

Definition 3.9.12. *Two matrices* A *and* B *are called* ***similar***, *if there is an invertible matrix* P *such that* $A = P^{-1}BP$.

Proposition 3.9.13. *Two matrices that represents the same linear application in different basis has the same rank.*
In particular, two similar matrices has the same rank.