Chapter 4

Systems of Linear Equations

This chapter focuses on solving systems of linear equations. You will learn methods such as Cramer's rule and the Jordan elimination method to find solutions. Solved exercises are included to help you practice and apply these techniques effectively.

4.1 Linear Systems

A linear equation in variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are constant real or complex numbers. The constant a_i is called the *coefficient* of x_i, and b is the *constant term* of the equation.

A *system of linear equations* (or *linear system*) is a finite collection of linear equations in the same variables. For instance, a linear system of m equations in n variables $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \qquad\qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

Solutions to Linear Systems

A solution of a linear system is a tuple $(s_1, s_2, \ldots, s_n)$ of numbers that satisfies all equations when substituted for $x_1, x_2, \ldots, x_n$. The set of all solutions is called the *solution set*.

Theorem 4.1.1. *Any system of linear equations has one of the following exclusive conclusions:*

1. *No solution,*

2. *A unique solution,*

3. *Infinitely many solutions.*

A linear system is said to be *consistent* if it has at least one solution, and *inconsistent* if it has no solution.

For example, consider the following three linear systems:

1.

$$\begin{cases} 2x_1 + x_2 = 3, \\ 2x_1 - x_2 = 0, \\ x_1 - 2x_2 = 4. \end{cases}$$

2.

$$\begin{cases} 2x_1 + x_2 = 3, \\ 2x_1 - x_2 = 5, \\ x_1 - 2x_2 = 4. \end{cases}$$

3.

$$\begin{cases} 2x_1 + x_2 = 3, \\ 4x_1 + 2x_2 = 6, \\ 6x_1 + 3x_2 = 9. \end{cases}$$

These systems have no solution, a unique solution, and infinitely many solutions, respectively.

4.2 Matrices of a Linear System

Augmented and Coefficient Matrices

The *augmented matrix* of the general linear system is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

The *coefficient matrix* is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

4.3 Solving Linear Systems

4.3.1 Solving Linear Systems Using Cramer's Rule

Definition 4.3.1. *Cramer's Rule is a method used to solve a system of* n *linear equations with* n *variables:*

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where $\mathbf{A}$ *is an* $n \times n$ *square matrix,* $\mathbf{x}$ *is the column vector of unknowns, and* $\mathbf{b}$ *is the column vector of constants.*

Theorem 4.3.2. *If* $\det(\mathbf{A}) \neq 0$*, the system has a unique solution given by:*

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}, \quad i = 1, 2, \ldots, n,$$

where $\mathbf{A}_i$ *is the matrix obtained by replacing the* i*-th column of* $\mathbf{A}$ *with* $\mathbf{b}$*.*

Condition: Cramer's Rule applies only when:

1. $\mathbf{A}$ is square ($n \times n$).

2. $\det(\mathbf{A}) \neq 0$.

Steps of Cramer's Rule

1. Compute $\det(\mathbf{A})$.

2. For each i, construct $\mathbf{A}_i$ by replacing the i-th column of $\mathbf{A}$ with $\mathbf{b}$.

3. Compute $\det(\mathbf{A}_i)$ for each i.

4. Solve for each x_i using:

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}.$$

Example 4.3.3. *Solve the system*

$$\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ x + 2y - z = 3. \end{cases}$$

1. *Write the coefficient matrix* $\mathbf{A}$*, the unknown vector* $\mathbf{x}$*, and the constant vector* $\mathbf{b}$*:*

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}.$$

2. *Compute* $\det(\mathbf{A})$*:*

$$\det(\mathbf{A}) = 1 \cdot \det \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

After computations, $\det(\mathbf{A}) = 7$*.*

3. *Replace each column of* $\mathbf{A}$ *with* $\mathbf{b}$ *to compute* $\mathbf{A}_1$*,* $\mathbf{A}_2$*, and* $\mathbf{A}_3$*:*

$$\mathbf{A}_1 = \begin{bmatrix} 6 & 1 & 1 \\ 3 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & -1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 1 & 6 \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

4. *Compute the determinants:*

$$\det(\mathbf{A}_1) = 9, \quad \det(\mathbf{A}_2) = 15, \quad \det(\mathbf{A}_3) = 18.$$

5. *Solve for* x, y, *and* z:

$$x = \frac{\det(\mathbf{A}_x)}{\det(\mathbf{A})} = \frac{9}{7}, \quad y = \frac{\det(\mathbf{A}_y)}{\det(\mathbf{A})} = \frac{15}{7}, \quad z = \frac{\det(\mathbf{A}_z)}{\det(\mathbf{A})} = \frac{18}{7}.$$

Hence, $x = \frac{9}{7}$, $y = \frac{15}{7}$, $z = \frac{18}{7}$.

4.3.2 Solving Linear Systems Using the Gauss-Jordan Method

The **Gauss-Jordan method** is an algorithm for solving linear systems of equations. It simplifies the augmented matrix of the system into its reduced row-echelon form (RREF), making the solution explicit.

Definition 4.3.4. *(Augmented Matrix) A system of linear equations can be written as an augmented matrix:*

$$[\mathbf{A} \mid \mathbf{b}],$$

where $\mathbf{A}$ *is the coefficient matrix and* $\mathbf{b}$ *is the column vector of constants.*

Definition 4.3.5. *(Row-Echelon Form (REF)) A matrix is in row-echelon form if:*

1. *All nonzero rows are above rows of all zeros.*

2. *The leading entry (pivot) of each nonzero row is to the right of the leading entry of the row above it.*

Definition 4.3.6. *(Reduced Row-Echelon Form (RREF)) A matrix is in reduced row-echelon form if, in addition to being in REF:*

1. *Each leading entry is 1.*

2. *Each leading 1 is the only nonzero entry in its column.*

Theorem 4.3.7. *(Uniqueness of RREF) Every matrix has a unique reduced row-echelon form, irrespective of the sequence of row operations used.*

Steps of the Gauss-Jordan Method

1. Write the system as an augmented matrix.

2. Use the following row operations to transform the matrix into RREF:

    - Swap two rows.
    - Multiply a row by a nonzero scalar.
    - Add or subtract a multiple of one row to/from another row.

3. Once in RREF, we have:

    - If every variable corresponds to a pivot column, the system has a unique solution.
    - If any row is inconsistent (e.g., $0 = 1$), the system has no solution.
    - If there are free variables, the system has infinitely many solutions.

Example 4.3.8. *Solve the system of equations:*

$$\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ x + 2y - z = 3. \end{cases}$$

1. *Write the augmented matrix:*

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -1 & 3 \end{array}\right].$$

2. *Use row operations to transform the matrix into RREF:*

    - *Step 1: Eliminate the first column below the pivot.*

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1,$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 1 & -2 & -3 \end{array}\right].$$

- *Step 2: Scale the second row.*

$$R_2 \to -\frac{1}{3}R_2,$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & \frac{1}{3} & 3 \\ 0 & 1 & -2 & -3 \end{array}\right].$$

- *Step 3: Eliminate the second column.*

$$R_1 \to R_1 - R_2, \quad R_3 \to R_3 - R_2,$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 3 \\ 0 & 1 & \frac{1}{3} & 3 \\ 0 & 0 & -\frac{7}{3} & -6 \end{array}\right].$$

- *Step 4: Scale the third row.*

$$R_3 \to -\frac{3}{7}R_3,$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 3 \\ 0 & 1 & \frac{1}{3} & 3 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

- *Step 5: Eliminate the third column.*

$$R_1 \to R_1 - \frac{2}{3}R_3, \quad R_2 \to R_2 - \frac{1}{3}R_3,$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

3. *The matrix is now in RREF. The solution is:*

$$x = -1, \quad y = 2, \quad z = 2.$$

Hence, $x = -1$*,* $y = 2$*,* $z = 2$*.*

Example 4.3.9. *Find all solutions for the linear system:*

$$\begin{cases} x_1 + 2x_2 - x_3 = 1, \\ 2x_1 + x_2 + 4x_3 = 2, \\ 3x_1 + 3x_2 + 4x_3 = 1. \end{cases}$$

Perform row operations:

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 6 & 0 \\ 0 & -3 & 7 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

The system is equivalent to:

$$x_1 = 7,$$

$$x_2 = -4,$$

$$x_3 = -2.$$

Example 4.3.10. *Solve the linear system:*

$$\begin{cases} x_1 - x_2 + x_3 - x_4 = 2, \\ x_1 - x_2 + x_3 + x_4 = 0, \\ 4x_1 - 4x_2 + 4x_3 = 4, \\ -2x_1 + 2x_2 - 2x_3 + x_4 = -3. \end{cases}$$

Perform row operations:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 4 & 0 & 4 \\ -2 & 2 & -2 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution is:

$$x_1 = 1 + x_2 - x_3,$$

$$x_4 = -1.$$

Variables x_2 and x_3 are free. Let $x_2 = c_1$, $x_3 = c_2$, where $c_1, c_2 \in \mathbb{R}$. Then:

$$x_1 = 1 + c_1 - c_2, \quad x_4 = -1.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$