University of science and technology Oran M-B
Department of Mathematics
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The 16^{th} May, 2024

Final exam of Algebra 2 (Duration 1h30)

Exercice 01:(08 points)

I. Let E and F be two $\mathbb{F}$- vector spaces, and f is a linear application from E to F.

1. State the definition of the kernel and the image of f ($ker f$ and $Im f$).
2. Give the definition of direct sum of two subspaces of an $\mathbb{F}$-vector space.

II. Let the following two sets

$$E_1 = \{(x, y, z) \in \mathbb{R}^3,\ x + y = z\} \text{ and } E_2 = \{(x, y, z) \in \mathbb{R}^3,\ x - y + 2z = 0 \wedge 2x + y + z = 0\}.$$

1. Prove that E_1 and E_2 are subspaces of $\mathbb{R}^3$.
2. Determine a basis for E_1 and a basis for E_2, then deduce their dimensions.
3. Prove that $\mathbb{R}^3 = E_1 \oplus E_2$.

Exercice 02:(08 points) Consider the following linear application f defined by

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R}^3 \\ (x, y, z) &\mapsto (2x + y, -3x - y + z, x - z) \end{aligned}$$

1. Write the matrix M associated to f in the canonical basis $B_0 = \{e_1(1, 0, 0), e_2(0, 1, 0), e_3(0, 0, 1)\}$ of $\mathbb{R}^3$.
2. Determine $Ker f$, then deduce the rank of the matrix M.
3. Is f an automorphism of $\mathbb{R}^3$? Justify your answer.
4. Let $u_1 = (1, -2, 1)$, $u_2 = (0, -1, 1)$ and $u_3 = (0, 0, 1)$.
   (a) Prove that $B_1 = \{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$.
   (b) The transition matrix P from the basis B_0 to the basis B_1 is given by $P = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Compute P^2, then deduce the matrix P^{-1}.
   (c) Write the matrix M' associated to f in the basis B_1.
   (d) Compute M^3, then prove that $M'^3 = O_{\mathcal{M}_3(\mathbb{R})}$, where $O_{\mathcal{M}_3(\mathbb{R})}$ is the null matrix.

Exercice 03:(04 points) Let the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

1. Compute A^2.
2. Prove that $A^2 = 2A + I_2$.
3. Deduce that A is invertible, and give its inverse A^{-1}.
4. Solve the system $AX = B$ of unknown $X \in \mathbb{R}^2$, for $B = (2, 1)^t$ and for $B = (1, 3)^t$.

Note: Exercise 2 will be counted as a replacement test for justified absences from tests 1 and 2.

Good luck