University of science and technology Oran M-B
Department of Mathematics
Faculty of mathematics and computer science
The 16^{th} January, 2025

Final exam of Algebra I (Duration 1h30)

Exercice 01:(05 points)
Let E be a set and A, B, C be three elements of $\mathcal{P}(E)$.

1. Prove that, if $A \cap B = A \cup B$, then $A = B$.
2. Does ($C \subset A \cup B$) imply ($C \subset A$ or $C \subset B$)? Justify.

Exercice 02:(08 points)
Let f be an application from $\mathbb{R}$ to $\mathbb{R}$ defined by:

$$f(x) = x^3 - x^2 - 9x + 9.$$

(I) Let the sets $S_1 = \{-3, 0, 1, 3\}$ and $S_2 = \{9\}$.

1. Determine $f(S_1)$. Deduce that f is not injective, justify.
2. Compute $f^{-1}(S_2)$.

(II) We define on $\mathbb{R}$ the relation $\mathcal{R}$ by:

$$\forall x, y \in \mathbb{R},\ x\mathcal{R}y \Leftrightarrow f(x) = f(y).$$

1. Prove that $\mathcal{R}$ is an equivalence relation on $\mathbb{R}$.
2. Discuss, based on the values of a, the number of elements in the equivalence class of a. Does $Cl(3) = Cl(-3)$?

Exercice 03:(07 points)

(I) Let α be a non-zero real parameter, we define on $\mathbb{R} - \{\alpha\}$ the binary operation $*$ by:

$$\forall x, y \in \mathbb{R} - \{\alpha\}, x * y = x + y - \frac{1}{\alpha}xy.$$

1. Prove that $\mathbb{R} - \{\alpha\}$ is an abelian group.
2. Let f be an application from $\mathbb{R} - \{\alpha\}$ to $(\mathbb{R} - \{0\})$ defined by:

$$f(x) = -\frac{1}{\alpha}x + 1.$$

   Prove that f is a group homomorphism from $(\mathbb{R} - \{\alpha\}, *)$ to $(\mathbb{R} - \{0\}, \bullet)$ (Where $\bullet$ is the usual binary operation).

(II) (Course Questions)
Let H and K be two subgroups of a group $(G, *)$, where $*$ is a binary operation on G:

1. Prove that $H \cap K$ is a subgroup of $(G, *)$.
2. Is $H \cup K$ a subgroup of $(G, *)$, justify.