Chapter 2 Direct Methods for Solving Linear Systems

Definitions

Definition 1:
A *linear equation* is an equation that can be expressed in the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Where $a_1, a_2, \ldots, a_n$ are coefficients, $x_1, x_2, \ldots, x_n$ are variables, and b is a constant.

Definition 2:
A *linear system of equations* consists of two or more linear equations involving the same set of variables.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

where $x_1, x_2, \ldots, x_n$ are the unknowns, $a_{11}, a_{12}, \ldots, a_{nn}$ are the coefficients of the system, and $b_1, b_2, \ldots, b_n$ are the constant terms.
There are three possible behaviors of a linear system: infinitely many solutions, a single unique solution $\left(x_1, x_2, \ldots, x_n\right)^T$, and no solution.
Understanding the possible behaviors of linear systems—infinitely many solutions, a single unique solution, and no solution—is crucial in both theoretical and practical contexts. Each behavior has distinct implications for how we approach solving systems of equations and understanding the relationships between variables. By identifying the nature of a linear system, we can apply the appropriate methods for analysis and solution.

Definition 3:
Triangular and *diagonal systems* are special cases of linear systems that offer the advantage of being very easy to solve. They form an important foundation for understanding and implementing more general methods for solving linear systems.

1) *Upper triangular system*, each equation has the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \qquad\quad a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ \qquad\qquad\qquad\qquad a_{nn}x_n = b_n \end{cases}$$

   Here, all coefficients below the main diagonal are zero ($\forall i, j = 1..n, i > j : a_{ij} = 0$).

2) *Lower triangular system*, the arrangement is inverted:

$$\begin{cases} \qquad\qquad\qquad\qquad a_{1n}x_n = b_1 \\ \qquad\quad a_{2(n-1)}x_{n-1} + a_{2n}x_n = b_2 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

   Where ($\forall i, j = 1..n, i < j : a_{ij} = 0$).

3) In a *diagonal system* of linear equations, the arrangement of the coefficients is such that all non-zero elements lie on the main diagonal of the coefficient matrix.

4) $\begin{cases} a_{11}x_1 = b_1 \\ a_{22}x_2 = b_2 \\ \ldots \ldots \ldots \ldots \\ a_{nn}x_n = b_n \end{cases}$

   Where ($\forall i, j = 1..n, i \neq j : a_{ij} = 0 \; and \;\; a_{ii} \neq 0$).

2.1 Remarks on Solving Triangular Systems

One of the fundamental principles in linear algebra is that any system of linear equations can be transformed into a triangular system. This is typically achieved through a process called ***Gaussian elimination*** or ***LU decomposition***. By manipulating the equations using row operations such as swapping rows, multiplying rows by non-zero constants, and adding multiples of one row to another one can systematically eliminate variables to achieve either an upper or lower triangular form.

2.1.1 Solving Upper Triangular System

To solve an upper triangular system, we use **back substitution**. The idea is to start from the last equation and substitute backward. The process is:

1) Solve for x_n:

$$x_n = \frac{b_n}{a_{nn}} \tag{10}$$

2) Substitute x_n into the previous-to-First equation and Solve for $\mathrm{i} = \mathrm{n} - 1 \ldots 1, x_i$:

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} \tag{11}$$

Example:

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 7 \\ 4x_2 + 2x_3 = 10 \\ 5x_3 = 15 \end{cases} \rightarrow \begin{cases} 2x_1 + 3x_2 + x_3 = 7 \\ 4x_2 + 2x_3 = 10 \\ \boldsymbol{x_3 = \frac{15}{5} = 3} \end{cases} \rightarrow \begin{cases} 2x_1 + 3x_2 + x_3 = 7 \\ \boldsymbol{x_2 = \frac{10 - 2(3)}{4} = 1} \\ \boldsymbol{x_3 = 3} \end{cases} \rightarrow \begin{cases} \boldsymbol{x_1 = \frac{7 - 3(1) - (3)}{2} = \frac{1}{2}} \\ \boldsymbol{x_2 = 1} \\ \boldsymbol{x_3 = 3} \end{cases}$$

The solution x is $\left(\frac{1}{2}, 1{,}3\right)^T$

2.1.1 Solving Lower Triangular System

To solve a lower triangular system, we use forward substitution. The steps are as follows:

1) Solve for x_1:

$$x_1 = \frac{b_1}{a_{11}} \tag{12}$$

2) Substitute x_1 into the second-to-last equation and Solve for $\mathrm{i} = 2 \ldots \mathrm{n}, x_i$:

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j}{a_{ii}} \tag{13}$$

Example:

$$\begin{cases} 2x_1 = 6 \\ 4x_1 + 3x_2 = 18 \\ 5x_1 - 2x_2 + 3x_3 = 7 \end{cases} \rightarrow \begin{cases} \boldsymbol{x_1 = \frac{6}{2} = 3} \\ 4x_1 + 3x_2 = 18 \\ 5x_1 - 2x_2 + 3x_3 = 7 \end{cases} \rightarrow \begin{cases} \boldsymbol{x_1 = 3} \\ \boldsymbol{x_2 = \frac{18 - 4(3)}{3} = 2} \\ 5x_1 - 2x_2 + 3x_3 = 7 \end{cases} \rightarrow \begin{cases} \boldsymbol{x_1 = 3} \\ \boldsymbol{x_2 = 2} \\ \boldsymbol{x_3 = \frac{7 - 5(3) + 2(2)}{3} = -\frac{4}{3}} \end{cases}$$

The solution x is $\left(3,2,-\frac{4}{3}\right)^T$

Triangular systems of equations provide a structured and efficient framework for solving linear equations. By transforming any system into a triangular form, we can leverage back substitution for upper triangular systems and forward substitution for lower triangular systems. Understanding these concepts is crucial for advanced numerical methods and applications in various fields, including engineering and computer science.

2.2 Gaussian Elimination Method

Gaussian elimination is a powerful and systematic method for solving systems of linear equations. The primary goal of this technique is to transform a rectangular system represented as AX= B into an equivalent upper triangular form, denoted as UX= C. In this representation, A is the coefficient matrix, X is the vector of variables, and B is the constant vector. The matrix U is an upper triangular matrix, and C is a modified constant vector resulting from the elimination process.

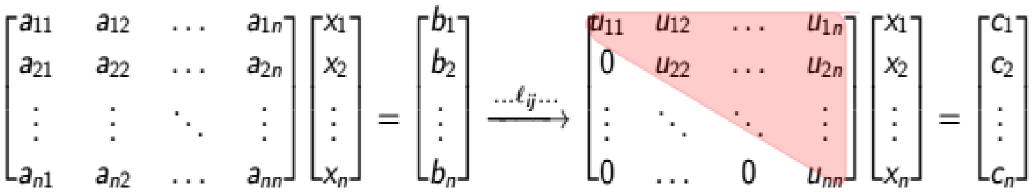

Figure 3. *Gaussian Elimination Method*

In summary, the Gaussian elimination process consists of two main steps: triangularization and resolution. First, we express a system of linear equations in the matrix form AX= B, combining the coefficient matrix A and constants vector B into an augmented matrix [A|B] The goal of triangularization is to transform this matrix into row echelon form, where all entries below the main diagonal are zeros. If a pivot is zero, we handle this by swapping with a non-zero row below it. After achieving row echelon form, the resolution step involves back substitution: starting from the last equation, we express each variable in terms of the others and substitute back up to find the values of all unknowns.

Step 1: Triangularization

1) **Writing the System in Matrix Form:** To begin, we express the system of linear equations in the form:$\boldsymbol{AX = B}$

$$A = \begin{pmatrix} a_{11}\, a_{12} \dots\, a_{1n} \\ a_{21}\, a_{22} \dots\, a_{2n} \\ \vdots \quad \vdots \quad\quad \vdots \\ a_{n2}\, a_{n2} \cdots\, a_{nn} \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{14}$$

Where:A is the coefficient matrix, X is the vector of unknowns, B is the constants vector.

2) **Formulating the Augmented Matrix**

Combine the matrix A and vector B into an augmented matrix denoted as $[A|B]$:

$$[A|B] = \left[\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n2} & \cdots & a_{nn} \end{matrix}\middle| \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix}\right] \tag{15}$$

3) Transforming to Row Echelon Form

The objective is to convert the augmented matrix into an upper triangular form (row echelon form), where all entries below the main diagonal are zeros.

$$[A|B] = \left[\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n2} & \cdots & a_{nn} \end{matrix}\middle| \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix}\right] \begin{matrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{matrix} \tag{16}$$

4) Using the Elimination Formula

For $i = 1..(n-1)$ each pivot a_{ii}, replace the entries below it using the formula:

$$For\ j = i + 1 \dots n,\ \ L_j = \ \ L_j - \frac{a_{ji}}{a_{ii}} \times \ \ L_i \tag{17}$$

Where:

- L_j is the j^{th}row,
- $L_i$is the i^{th}row (the pivot row),
- a_{ji} is the element below the pivot to reduce it to zero at position (j, i)
- a_{ii} is the pivot element at position (i, i).

5) Handling Zero Pivots

- **Identifying a Zero Pivot**

If the pivot element a_{ii} is zero, we cannot proceed with elimination directly, as division by zero is undefined.

- **Row Swapping**

To resolve this issue, look for a non-zero entry in the same column below the current pivot row. If found, swap the current row with the row containing the non-zero entry: $L_i \leftrightarrow L_k$
Where L_k is the row with the non-zero element: $i < K \le n,\ a_{ki} \neq 0$.
After swapping, continue using the elimination formula as before.

6) Repeat the elimination process for each column until the matrix is in row echelon form

Step 2: Resolution

After triangularization, we obtain the following matrix system:

$$U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & a_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}, C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Next, we transform the new system UX=C into a system of linear equations to solve for X, which will be the same solution as for the original system AX=B.

$$\begin{cases} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n = c_1 \\ \quad u_{22}x_2 + \cdots + u_{2n}x_n = c_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \quad u_{nn}x_n = c_n \end{cases} \tag{18}$$

1) **Back Substitution**
   - Solve for x_n:

$$x_n = \frac{c_n}{u_{nn}} \tag{19}$$

   - Substitute x_n into the previous-to-First equation and Solve for $\mathrm{i} = \mathrm{n} - 1 \dots 1, x_i$:

$$x_i = \frac{c_i - \sum_{j=i+1}^{n} u_{ij}x_j}{u_{ii}} \tag{20}$$

By transforming the system UX=C into a system of linear equations and solving through back substitution, you can effectively find the solution vector X, similar to how you would have solved AX=B. This process maintains the same structure and approach, ensuring consistency in solving linear systems.

Example:

Consider the following system of equations:

$$\begin{cases} 2x + 3y + z = 1 \\ 4x + y + 2z = 2 \\ -2x + 5y + 2z = 3 \end{cases}$$

Step 1: Write the Augmented Matrix
First, we form the augmented matrix [A|B]:

$$\begin{array}{c} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 4 & 1 & 2 & 2 \\ -2 & 5 & 2 & 3 \end{array}\right] \end{array} \begin{array}{l} L_1 \\ L_2 \\ L_3 \end{array}$$

Step 2: Triangularization

For i=1 to (n-1)=2

1. $i = 1$ **pivot** $a_{11} = 2 \neq 0$

Eliminate the first column below the pivot by following Eq. (17):

$$L_j = L_j - \frac{a_{ji}}{a_{ii}} \times L_i$$

- For $L_2 = L_2 - \frac{4}{2}L_1 = L_2 - 2L_1$
- For $L_3 = L_3 - \frac{(-2)}{2}L_1 = L_3 + L_1$

The calculations for L_2 and L_3 are as follows:

$L_2 = (4\ \ 1\ \ 2|2) - 2(\ 2\ 3\ 1|1) = (4\ \ 1\ \ 2|2) - (\ 4\ 6\ 2|2) = (0\ \ -5\ \ 0|0)$
$L_3 = (-2\ \ 5\ 2|3) + (\ 2\ 3\ 1|1) = (\ 0\ 8\ 3|4)$

The augmented matrix is now:

$$\begin{bmatrix} 2 & 3 & 1 & | & 1 \\ 0 & -5 & 0 & | & 0 \\ 0 & 8 & 3 & | & 4 \end{bmatrix} \begin{matrix} L_1 \\ L_2 \\ L_3 \end{matrix}$$

2. $i = 2$ **pivot** $a_{22} = -5 \neq 0$

Eliminate the first column below the next pivot

- For $L_3 = L_3 - \frac{8}{(-5)} L_1 = L_3 + L_1$

$$L_3 = (0\ \ -8\ \ 3|4) + \frac{8}{5}(0\ \ -5\ \ 0|0) = (0\ \ 0\ \ 3|4)$$

Finally, the augmented matrix is now:

$$\begin{bmatrix} 2 & 3 & 1 & | & 1 \\ 0 & -5 & 0 & | & 0 \\ 0 & 0 & 3 & | & 4 \end{bmatrix}$$

Step 3: Back Substitution

From the final augmented matrix, we can write the equations:

$$\begin{cases} 2x + 3y + z = 1 \\ -5y = 0 \\ 3z = 4 \end{cases} \rightarrow \begin{cases} x = -\dfrac{1}{6} \\ y = 0 \\ z = \dfrac{4}{3} \end{cases}$$

Final Solution

The solution to the system AX=B is: $x = \left(-\frac{1}{6}, 0, \frac{4}{3}\right)^T$

Exercise

Consider the following system of linear equations represented by:

$$\begin{cases} 2y + z = 4 \\ x + 3y + 2z = 5 \\ 2x + y + z = 3 \end{cases}$$

1. Form the augmented matrix [A|B] and perform Gaussian elimination.
2. Then, deduce the determinant of A.
   NB: $\mathbf{det}(\boldsymbol{A}) = (-\mathbf{1})^{\boldsymbol{p}}.\prod_{i=1}^{n} \boldsymbol{u}_{ii}$
   Where *p* is the number of permutations performed during the triangularization of *A* and U is an upper triangular matrix.

2.3 Matrix Interpretation of Gaussian Elimination: LU Factorization

LU Factorization is a method used to decompose a given matrix A into two matrices: a lower triangular matrix L and an upper triangular matrix U. This decomposition is particularly useful for solving systems of linear equations, calculating determinants, and performing matrix inversions efficiently (as shown in Figure.4).

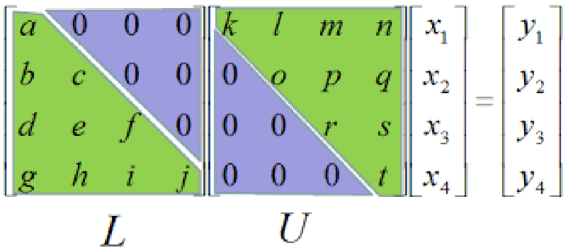

Figure 4. *Representation of LU Factorization*

For a matrix A to be decomposed into its LU factorization, it must satisfy several conditions: First, A must be a square matrix. Second, it must be non-singular, meaning that its determinant is non-zero (det(A)≠0). Lastly, if zero pivots are encountered during the elimination process, the use of permutation matrices may be necessary to enable the factorization. These conditions collectively ensure that the LU factorization can be performed without issues.

Steps to Obtain LU Factorization from Gaussian Elimination:

1) **Initial Matrix**: Consider a square matrix A of size n×n.
2) **Form of the Decomposition**: We seek to express:

A= LU

$$A = \begin{pmatrix} a_{11} a_{12} \dots a_{1n} \\ a_{21} a_{22} \dots a_{2n} \\ \vdots \quad \vdots \qquad \vdots \\ a_{n2} a_{n2} \cdots \ a_{nn} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & \dots 0 \\ l_{21} & 1 & \dots 0 \\ \vdots & \vdots & \vdots \\ l_{n2} & l_{n2} & \cdots 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} u_{11} & u_{12} \dots & u_{1n} \\ 0 & u_{22} \dots & u_{2n} \\ \vdots & \vdots & \vdots \\ 0 & 0 \ \cdots & u_{nn} \end{pmatrix}}_{U} \tag{21}$$

Where L is a lower triangular matrix with ones on the diagonal and U is an upper triangular matrix.

3) **Gaussian Elimination**:
   Apply Gaussian elimination to transform A into an upper triangular matrix U.
   At each step of elimination, as you eliminate the entries below the diagonal of A, note the multipliers used to zero out the elements in the current column. These multipliers will form the elements of L.
4) **Constructing L and U**:
   For each non-zero element a_{ij} used for elimination, record the multiplier $m_{ij} = \frac{a_{ij}}{a_{jj}}$ and place this multiplier in the position $(i, j)$of matrix L.
   The matrix U will be the resulting matrix after all elimination steps.

When we have a linear system represented by the matrix equation: AX=B And we have a LU decomposition of matrix A, meaning A = LU, we can substitute LU for A in the original equation, as shown in Eq. (22):

$$AX = B \rightarrow (LU)X = B \rightarrow L(UX) = B \rightarrow LY = B$$

$$where \begin{cases} LY = B & \quad is\ the\ first\ system\ to\ solve \\ UX = Y & is\ the\ second\ system\ to\ solve \end{cases} \tag{22}$$

By introducing a new variable Y, we can break down the problem into two simpler triangular systems:

1. To solve for Y, we start by defining Y=UX. This transforms our equation into LY=B. Given that L is a lower triangular matrix, we can apply forward substitution to efficiently solve for Y.
2. After obtaining Y, we proceed to solve the equation UX= Y. Since U is an upper triangular matrix, we can efficiently find X using backward substitution.

By factoring A into LU, we transform a potentially complex linear system into two simpler triangular systems, which can be solved efficiently using forward and backward substitution.

Example:

Consider the following matrix A:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l12 & 1 & 0 \\ l13 & l32 & 1 \end{bmatrix} * \begin{bmatrix} u11 & u12 & u13 \\ 0 & u22 & u23 \\ 0 & 0 & u33 \end{bmatrix}$$

Using Gaussian elimination on A

For i=1 to (n-1) =2

- **Calculate i^{th} column of L**
- **Calculate i^{th} row of U**

1) For L: $i = 1 \; and \; k = 2..n, \;\; l_{ki} = \frac{a_{ki}^{(i-1)}}{a_{ii}^{(i-1)}}$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ l12 = \dfrac{a^{(0)}{}_{12}}{a^{(0)}{}_{11}} = \dfrac{4}{1} = 4 & 1 & 0 \\ l13 = \dfrac{a^{(0)}{}_{13}}{a^{(0)}{}_{11}} = \dfrac{3}{1} = 3 & l32 & 1 \end{bmatrix}$$

For U: Eliminate Entries Below the Pivot i=1: $a_{11} = 1 \neq 0$

$$For \; j = i + 1 \ldots n, \;\; L_j = \;\; L_j - \frac{a_{ji}}{a_{ii}} \times \;\; L_i$$

$$A^{(0)} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{matrix} L_2 = L_2 - \frac{4}{1} L_1 \\ L_3 = L_{32} - \frac{3}{1} L_1 \end{matrix} \rightarrow A^{(1)} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix}$$

2) For L: $i = 2 \; and \; k = 2..n, \;\; l_{ki} = \frac{a_{ki}^{(k-1)}}{a_{ii}^{(k-1)}}$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & l32 = \dfrac{a^{(1)}{}_{32}}{a^{(1)}{}_{22}} = \dfrac{2}{-1} = -2 & 1 \end{bmatrix}$$

Eliminate Entries Below the Pivot 2: $a_{22} = -1 \neq 0$

$$A^{(1)} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix} L_3 = L_{32} - \frac{2}{(-1)} L_1 = L_3 + 2L_1 \to A^{(2)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} = U$$

The obtained matrix is the upper triangular matrix U

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

Exercise

Consider the following system of linear equations::

$$\begin{cases} 2x + 3y + z = 1 \\ 4x + 7y + 2z = 2 \\ 6x + 18y + 5z = 3 \end{cases}$$

1. Perform LU decomposition of the coefficient matrix A
2. Solve the system using the LU decomposition method.
3. Then, deduce the determinant of A.