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Mathematical Representation of Simple Harmonic Motion

Exercise 1

In an engine, a piston oscillates with simple harmonic motion so that its position varies according to the expression $x = (5.00\ cm)\cos(2.00t + \pi/6)$, where x is in centimeters and t is in seconds. At $t = 0$, find :

a) the position of the piston,
b) its velocity,
c) its acceleration,
d) the period and amplitude of the motion.

(a) $x = (5.00\text{ cm})\cos\left(2t + \frac{\pi}{6}\right)$ At $t = 0$, $x = (5.00\text{ cm})\cos\left(\frac{\pi}{6}\right) = \boxed{4.33\text{ cm}}$

(b) $v = \frac{dx}{dt} = -(10.0\text{ cm/s})\sin\left(2t + \frac{\pi}{6}\right)$ At $t = 0$, $v = \boxed{-5.00\text{ cm/s}}$

(c) $a = \frac{dv}{dt} = -(20.0\text{ cm/s}^2)\cos\left(2t + \frac{\pi}{6}\right)$ At $t = 0$, $a = \boxed{-17.3\text{ cm/s}^2}$

(d) $A = \boxed{5.00\text{ cm}}$ and $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \boxed{3.14\text{ s}}$

Exercise 2

The position of a particle is given by the expression
$x = (4.00\ m)\cos(3.00\pi t + \pi)$, where x is in meters and t is in seconds.
Determine

a) the frequency and period of the motion,
b) the amplitude of the motion,
c) the phase constant,
d) the position of the particle at $t = 0.250\ s$.

$x = (4.00\text{ m})\cos(3.00\pi t + \pi)$ Compare this with $x = A\cos(\omega t + \phi)$ to find

(a) $\omega = 2\pi f = 3.00\pi$

or $\boxed{f = 1.50\text{ Hz}}$ $T = \frac{1}{f} = \boxed{0.667\text{ s}}$

(b) $A = \boxed{4.00\text{ m}}$

(c) $\phi = \boxed{\pi\text{ rad}}$

(d) $x(t = 0.250\text{ s}) = (4.00\text{ m})\cos(1.75\pi) = \boxed{2.83\text{ m}}$

Exercise 3

a) A hanging spring stretches by $35.0\ cm$ when an object of mass $450\ g$ is hung on it at rest. In this situation, we define its position as $x = 0$. The object is pulled down an additional $18.0\ cm$ and released from rest to oscillate without friction. What is its position x at a time $84.4\ s$ later?

b) **What If?** A hanging spring stretches by $35.5\ cm$ when an object of mass $440\ g$ is hung on it at rest. We define this new position as $x = 0$. This object is also pulled down an additional $18.0\ cm$ and released from rest to oscillate without friction. Find its position $84.4\ s$ later.

c) Find the distance traveled by the vibrating object in part (a).

d) Find the distance traveled by the object in part (b).

e) Why are the answers to (c) and (d) different by such a large percentage when the data are so similar? Does this circumstance reveal a fundamental difficulty in calculating the future?

(a) The spring constant of this spring is

$$k = \frac{F}{x} = \frac{0.45\ \text{kg}\ 9.8\ \text{m/s}^2}{0.35\ \text{m}} = 12.6\ \text{N/m}$$

we take the x-axis pointing downward, so $\phi = 0$

$$x = A\cos\omega t = 18.0\ \text{cm}\cos\sqrt{\frac{12.6\ \text{kg}}{0.45\ \text{kg}\cdot\text{s}^2}}84.4\ \text{s} = 18.0\ \text{cm}\cos 446.6\ \text{rad} = \boxed{15.8\ \text{cm}}$$

(d) Now $446.6\ \text{rad} = 71\times 2\pi + 0.497\ \text{rad}$. In each cycle the object moves $4(18) = 72\ \text{cm}$, so it has moved $71(72\ \text{cm}) + (18 - 15.8)\ \text{cm} = \boxed{51.1\ \text{m}}$.

(b) By the same steps, $k = \dfrac{0.44\ \text{kg}\ 9.8\ \text{m/s}^2}{0.355\ \text{m}} = 12.1\ \text{N/m}$

$$x = A\cos\sqrt{\frac{k}{m}}t = 18.0\ \text{cm}\cos\sqrt{\frac{12.1}{0.44}}84.4 = 18.0\ \text{cm}\ \cos 443.5\ \text{rad} = \boxed{-15.9\ \text{cm}}$$

(e) $443.5\ \text{rad} = 70(2\pi) + 3.62\ \text{rad}$

$$\text{Distance moved} = 70(72\ \text{cm}) + 18 + 15.9\ \text{cm} = \boxed{50.7\ \text{m}}$$

(c) The answers to (d) and (e) are not very different given the difference in the data about the two vibrating systems. But when we ask about details of the future, the imprecision in our knowledge about the present makes it impossible to make precise predictions. The two oscillations start out in phase but get completely out of phase.

Exercise 4

A particle moving along the x axis in simple harmonic motion starts from its equilibrium position, the origin, at $t = 0$ and moves to the right. The amplitude of its motion is $2.00\ cm$, and the frequency is $1.50\ Hz$.

a) Show that the position of the particle is given by $x = (2.00\ m)\sin(3.00\pi t)$

b) Determine the maximum speed and the earliest time ($t > 0$) at which the particle has this speed,

c) Determine the maximum acceleration and the earliest time ($t > 0$) at which the particle has this acceleration,

d) Determine the total distance traveled between $t = 0$ and $t = 1.00$ s.

(a) At $t = 0$, $x = 0$ and v is positive (to the right). Therefore, this situation corresponds to $x = A\sin\omega t$

and $v = v_i \cos\omega t$

Since $f = 1.50$ Hz, $\omega = 2\pi f = 3.00\pi$

Also, $A = 2.00$ cm, so that $\boxed{x = (2.00\text{ cm})\sin 3.00\pi t}$

(b) $v_{\max} = v_i = A\omega = 2.00(3.00\pi) = 6.00\pi\text{ cm/s} = \boxed{18.8\text{ cm/s}}$

The particle has this speed at $t = 0$ and next at $t = \frac{T}{2} = \boxed{\frac{1}{3}\text{ s}}$

(c) $a_{\max} = A\omega^2 = 2.00(3.00\pi)^2 = 18.0\pi^2\text{ cm/s}^2 = \boxed{178\text{ cm/s}^2}$

This positive value of acceleration first occurs at $t = \frac{3}{4}T = \boxed{0.500\text{ s}}$

(d) Since $T = \frac{2}{3}$ s and $A = 2.00$ cm, the particle will travel 8.00 cm in this time.

Hence, in 1.00 s$\left(= \frac{3}{2}T\right)$, the particle will travel $8.00\text{ cm} + 4.00\text{ cm} = \boxed{12.0\text{ cm}}$.

Exercise 5

A piston in a gasoline engine is in simple harmonic motion. If the extremes of its position relative to its center point are $\mp 5.00\ cm$, find the maximum velocity and acceleration of the piston when the engine is running at the rate of 3 600 rev/min.

$x = A\cos\omega t$ $\quad A = 0.05$ m $\quad v = -A\omega\sin\omega t$ $\quad a = -A\omega^2\cos\omega t$

If $f = 3\,600\text{ rev/min} = 60\text{ Hz}$, then $\omega = 120\pi\text{ s}^{-1}$

$v_{\max} = 0.05(120\pi)\text{ m/s} = \boxed{18.8\text{ m/s}}$ $\quad a_{\max} = 0.05(120\pi)^2\text{ m/s}^2 = \boxed{7.11\text{ km/s}^2}$

Exercise 6

A 0.500-kg object attached to a spring with a force constant of 8.00 N/m vibrates in simple harmonic motion with an amplitude of 10.0 cm. Calculate (a) the maximum value of its speed and acceleration, (b) the speed and acceleration when the object is 6.00 cm from the equilibrium position, and (c) the time interval required for the object to move from $x = 0$ to $x = 8.00$ cm.

(a) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8.00\ \text{N/m}}{0.500\ \text{kg}}} = 4.00\ \text{s}^{-1}$ so position is given by $x = 10.0\sin(4.00t)$ cm.

From this we find that $v = 40.0\cos(4.00t)$ cm/s $\quad v_{max} = \boxed{40.0\ \text{cm/s}}$

$a = -160\sin(4.00t)\ \text{cm/s}^2 \quad a_{max} = \boxed{160\ \text{cm/s}^2}$.

(b) $t = \left(\frac{1}{4.00}\right)\sin^{-1}\left(\frac{x}{10.0}\right)$ and when $x = 6.00$ cm, $t = 0.161$ s.

We find $v = 40.0\cos[4.00(0.161)] = \boxed{32.0\ \text{cm/s}}$

$a = -160\sin[4.00(0.161)] = \boxed{-96.0\ \text{cm/s}^2}$.

(c) Using $t = \left(\frac{1}{4.00}\right)\sin^{-1}\left(\frac{x}{10.0}\right)$

when $x = 0$, $t = 0$ and when $x = 8.00$ cm, $t = 0.232$ s.

Therefore, $\Delta t = \boxed{0.232\ \text{s}}$.

Exercise 7

A 1.00-kg glider attached to a spring with a force constant of 25.0 N/m oscillates on a horizontal, frictionless air track. At $t = 0$ the glider is released from rest at $x = -3.00$ cm. (That is, the spring is compressed by 3.00 cm.) Find (a) the period of its motion, (b) the maximum values of its speed and acceleration, and (c) the position, velocity, and acceleration as functions of time.

Solution :

$m = 1.00$ kg, $k = 25.0$ N/m, and $A = 3.00$ cm. At $t = 0$, $x = -3.00$ cm

(a) $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{25.0}{1.00}} = 5.00 \text{ rad/s}$

so that, $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{5.00} = \boxed{1.26 \text{ s}}$

(b) $v_{max} = A\omega = 3.00 \times 10^{-2} \text{ m}(5.00 \text{ rad/s}) = \boxed{0.150 \text{ m/s}}$

$a_{max} = A\omega^2 = 3.00 \times 10^{-2} \text{ m}(5.00 \text{ rad/s})^2 = \boxed{0.750 \text{ m/s}^2}$

(c) Because $x = -3.00$ cm and $v = 0$ at $t = 0$, the required solution is $x = -A\cos\omega t$

or $\boxed{x = -3.00\cos(5.00t) \text{ cm}}$

$v = \dfrac{dx}{dt} = \boxed{15.0\sin(5.00t) \text{ cm/s}}$

$a = \dfrac{dv}{dt} = \boxed{75.0\cos(5.00t) \text{ cm/s}^2}$

Exercise 8

A 1.00-kg object is attached to a horizontal spring. The spring is initially stretched by 0.100 m, and the object is released from rest there. It proceeds to move without friction. The next time the speed of the object is zero is 0.500 s later. What is the maximum speed of the object?

Solution:

The 0.500 s must elapse between one turning point and the other. Thus the period is 1.00 s.

$$\omega = \frac{2\pi}{T} = 6.28/\text{s}$$

and $v_{\text{max}} = \omega A = (6.28/\text{s})(0.100 \text{ m}) = \boxed{0.628 \text{ m/s}}$.

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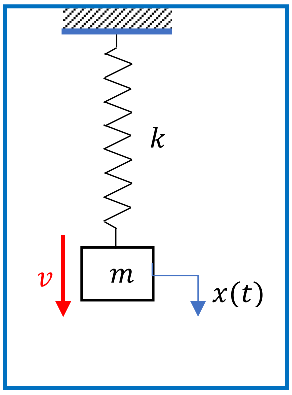
Exercise 9

A particle that hangs from a spring oscillates with an angular frequency ω. The spring is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car) as the car descends at a constant speed v. The car then stops suddenly. (a) With what amplitude does the particle oscillate? (b) What is the equation of motion for the particle? (Choose the upward direction to be positive.)

Solution :

At the instant the cabin stops, which is the initial instant, the mass is in static equilibrium position($x_0 = 0$) and the particle has a speed ($v_0 = v$).
The equation of motion $x(t) = A\cos(\omega t + \varphi)$

a) $\begin{cases} x_0 = A\cos\varphi \\ v_0 = -\omega A\sin\varphi \end{cases} \Rightarrow \begin{cases} A = \sqrt{\left(\frac{v_0}{\omega}\right)^2 + x_0^2} \\ \varphi = \tan^{-1}\left(-\frac{v_0}{\omega x_0}\right) \end{cases}$



$$A = \sqrt{\left(\frac{v_0}{\omega}\right)^2 + x_0^2} = \sqrt{\left(\frac{v}{\omega}\right)^2 + 0} = \frac{v}{\omega}$$

$$x_0 = A\cos\varphi = 0 \quad \Rightarrow \quad \varphi = \pm\frac{\pi}{2}$$

$$v_0 = -\omega A\sin\varphi = v > 0 \Rightarrow \quad \varphi = -\frac{\pi}{2}$$

b) $x(t) = \left(\frac{v}{\omega}\right)\cos\left(\omega t - \frac{\pi}{2}\right) = \cos\omega\, t\cos\frac{\pi}{2} - \left(\frac{v}{\omega}\right)\sin\omega t\sin\frac{\pi}{2}$

$$x(t) = -\left(\frac{v}{\omega}\right)\sin\omega t$$