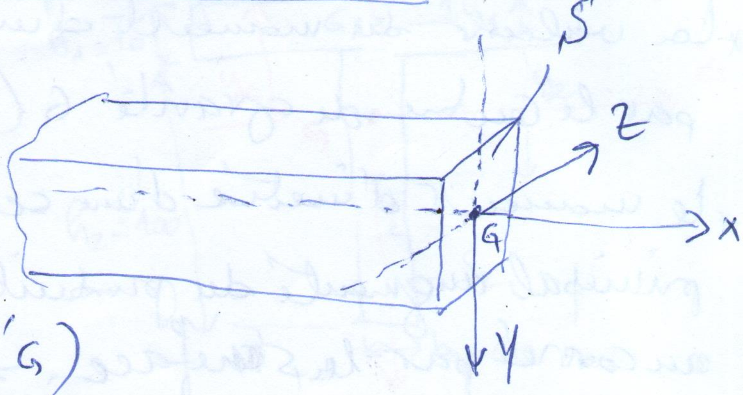
Moment quadratique :
moments d'inertie d'une section

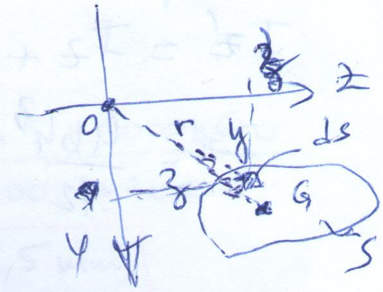
Soit une poutre :

(x) : axe longitudinal

(y) et (z) : axes principaux de la section S (passent par le centre de gravité G)



Soit une section (S) dans le système d'axes y, z



les moments d'inertie quadratiques :

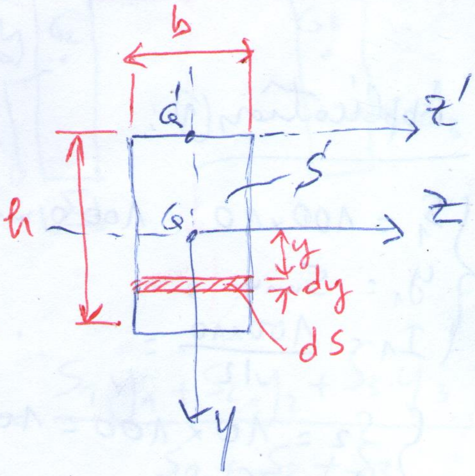
$$\begin{cases} I_y = \int_S z^2 ds \\ I_z = \int_S y^2 ds \end{cases}$$

le moment d'inertie polaire :

$$I_O = \int_S r^2 ds = \int_S (y^2 + z^2)\, ds = \overbrace{\int_S y^2 ds}^{I_z} + \overbrace{\int_S z^2 ds}^{I_y} = I_y + I_z$$

$$\boxed{I_O = I_y + I_z}$$

Exemple :



% z : $ds = b.dy$

$$I_z = \int_S y^2 ds = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b dy = b \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy = b \left[\frac{y^3}{3}\right]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$I_z = \frac{b}{3}\left[\left(\frac{h}{2}\right)^3 - \left(-\frac{h}{2}\right)^3\right] = \frac{b}{3}\left[\frac{h^3}{8} + \frac{h^3}{8}\right] = \frac{b}{3} \cdot \frac{h^3}{4}$$

$$\boxed{I_z = \frac{bh^3}{12}}$$

% z' : $I_{z'} = \int_S y^2 ds = \int_0^h y^2 b.dy = b \int_0^h y^2 dy = b \left(\frac{y^3}{3}\right)_0^h = \frac{bh^3}{3}$

$$\boxed{I_{z'} = \frac{bh^3}{3}} \Rightarrow I_z < I_{z'}$$

théorème des axes // :

$$\boxed{I = I_G + \Delta^2 . S}$$

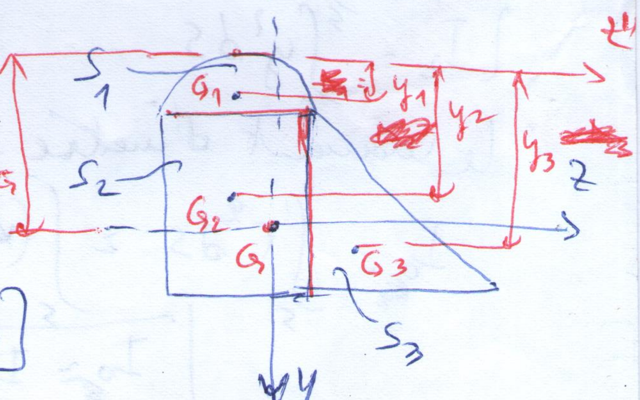
* La valeur du moment d'inertie est minimale / axe passant par le centre de gravité G (axe principale de la section)
* le moment d'inertie d'une section / ~~au~~ axe est celui // l'axe principal augmenté du produit de la distance (séparant les axes) au carré par la surface. →

$$I_{z'} = I_z + \Delta^2 S = \frac{bh^3}{12} + \left(\frac{h}{2}\right)^2 (bh) = \frac{bh^3}{12} + \frac{bh^3}{4} = \frac{bh^3}{12} + \frac{3bh^3}{12}$$

$$I_{z'} = \frac{4bh^3}{12} = \frac{bh^3}{3} \rightarrow \boxed{I_{z'} = \frac{bh^3}{3}}$$

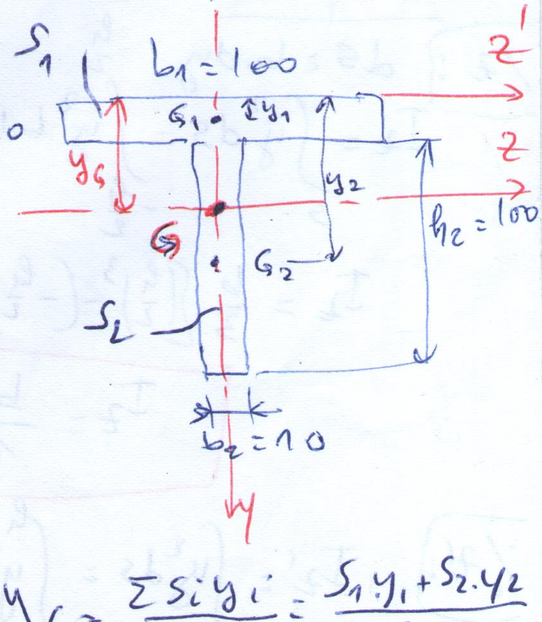
Surface Composée :

$$\boxed{I_z = \sum_{i=1}^{n} \left[I_{G_i} + \underbrace{(y_i - y_G)^2}_{\Delta_i^2} S_i \right]}$$



$$I_z = \left\{ I_{G_1} + \Delta_1^2 S_1 \right\} + \left\{ I_{G_2} + \Delta_2^2 S_2 \right\} + \left\{ I_{G_3} + \Delta_3^2 S_3 \right\}$$

Application (1) :



$$\begin{cases} S_1 = 100 \times 10 = 1000 \text{ mm}^2 \\ y_1 = 5 \text{ mm} \\ I_1 = \frac{100 \times 10^3}{12} = \quad . \quad \text{mm}^4 \end{cases}$$

$$\begin{cases} S_2 = 10 \times 100 = 1000 \text{ mm}^2 \\ y_2 = 60 \text{ mm} \\ I_2 = \frac{10 \times 100^3}{12} = \quad . \quad \text{mm}^4 \end{cases}$$

$$I_z = \left\{ I_1 + (y_1 - y_G)^2 . S_1 \right\} + \left\{ I_2 + (y_2 - y_G)^2 . S_2 \right\}$$

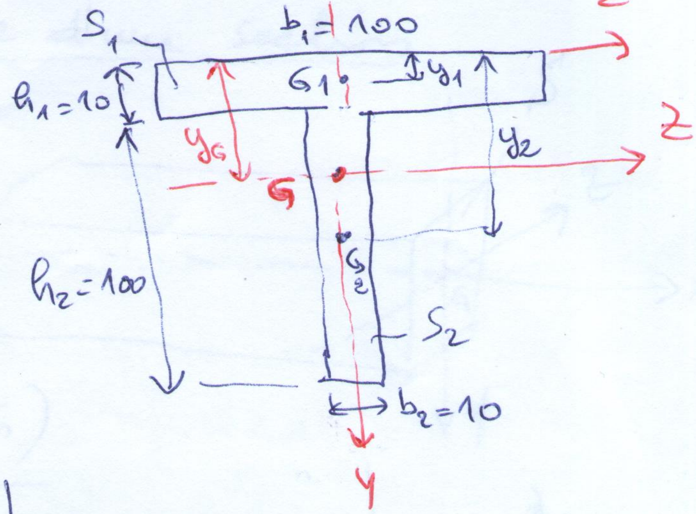
$$I_z = \quad . \quad \text{mm}^4$$

$$y_G = \frac{\sum S_i y_i}{\sum S_i} = \frac{S_1 . y_1 + S_2 . y_2}{S_1 + S_2}$$

$$y_G = 32{,}5 \text{ mm}$$

Applications _ Moments d'inertie

1) La position du centre G :



$$y_G = \frac{\sum_i S_i y_i}{\sum S_i} = \frac{S_1 \cdot y_1 + S_2 \cdot y_2}{S_1 + S_2}$$

$$\begin{cases} S_1 = \overset{b_1}{100} \times \overset{h_1}{10} = 1000\ mm^2 \\ y_1 = 5\ mm = \frac{h_1}{2} \\ I_1 = \frac{\overset{b_1}{100} \times \overset{h_1^3}{10^3}}{12} = 8333{,}33\ mm^4 \end{cases}$$

$$\begin{cases} S_2 = \overset{b_2}{10} \times \overset{h_2}{100} = 1000\ mm^2 \\ y_2 = \overset{h_1}{10} + \overset{h_2/2}{50} = 60\ mm \\ I_2 = \frac{\overset{b_2}{10} \times \overset{h_2^3}{100^3}}{12} = 833\,333{,}33\ mm^4 \end{cases}$$

$$y_G = \frac{1000 \times 5 + 1000 \times 60}{1000 + 1000}$$

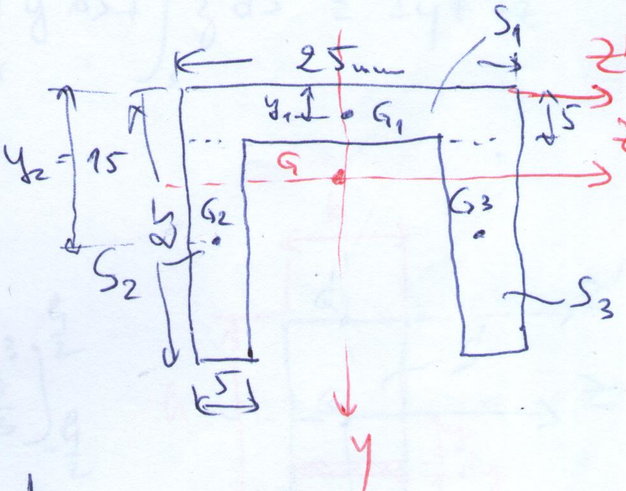
$$\boxed{y_G = 32{,}5\ mm}$$

$$I_Z = \sum_i \left(I_i + (y_i - y_G)^2 \cdot S_i\right) = \left[I_1 + (y_1 - y_G)^2 \times S_1\right] + \left[I_2 + (y_2 - y_G)^2 \cdot S_2\right]$$

$$I_Z = \quad . \quad mm^4$$

2)

$$\begin{cases} S_1 = 25 \times 5 = 125\ mm^2 \\ y_1 = \frac{5}{2} = 2{,}5\ mm \\ I_1 = \frac{25 \times 5^3}{12} = 260{,}42\ mm^4 \end{cases}$$



$$\begin{cases} S_2 = S_3 = 20 \times 5 = 100\ mm^2 \\ y_2 = y_3 = 15\ mm \\ I_2 = I_3 = \frac{5 \times 20^3}{12} = 3333{,}33\ mm^4 \end{cases}$$

$$y_G = \frac{S_1 \cdot y_1 + S_2 \cdot y_2 + S_3 \cdot y_3}{S_1 + S_2 + S_3}$$

$$\boxed{y_G = 10{,}2\ mm}$$

$$I_Z = \left[I_1 + (y_1 - y_G)^2 \cdot S_1\right] + \left[I_2 + (y_2 - y_G)^2 \cdot S_2\right] + \left[I_3 + (y_3 - y_G)^2 \cdot S_3\right]$$

$$I_Z = \quad . \quad mm^4$$

(Application pour le TP flexion)